College of Staten Island
Department of Mathematics

MTH 230/231 Calculus I Fall 2013 Common Final

NAME: Solutions

Each part of each question is worth **4 points**.

SHOW YOUR WORK—OTHERWISE THERE IS NO CREDIT!

Question	Possible Points	Earned Points
1	4	
2a	4	
2b	4	
2c	4	
3a	4	
3b	4	
3c	4	
3d	4	
4	4	
5	4	
6a	4	
6b	4	
6c	4	
6d	4	
7a	4	
7b	4	
8	4	
9a	4	
9b	4	
9c	4	
9d	4	
10a	4	
10b	4	
10c	4	
11	4	

1. Let f(x) be a function. Carefully define f ′ (x), the derivative of f(x).

f ′ (x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Graphically (i.e. geometrically) what does the derivative of f(x) measure?

Answer: the slope of the tangent line to the graph of f(x)

2. Let f(x) = $\frac{\sin(x)}{x}$. This function will be used in all parts of this question 2.

a. Use L'Hopital's rule to compute: $\lim_{x \to 0} \frac{\sin(x)}{x} =$ ~~lim~~ 1

Show your computation: $= \lim_{x \to 0} \frac{\cos(x)}{1} = 1$

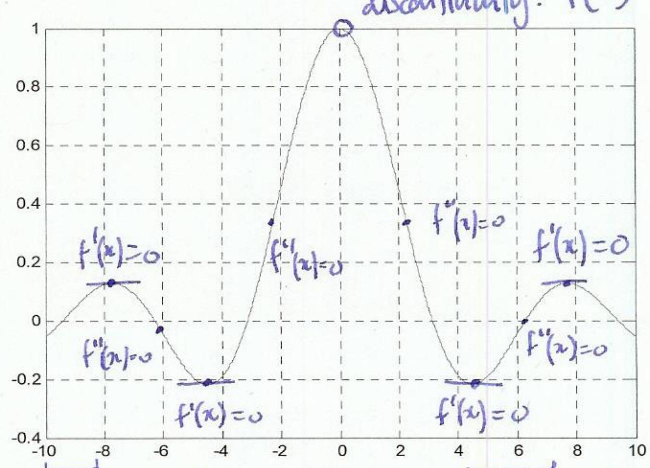
b. Use the "squeeze theorem" to compute $\lim_{x \to \infty} \frac{\sin(x)}{x} =$ 0

Hint: Look carefully at this limit.
Show your computation: $|\sin(x)| \le 1$ so $-\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$\lim_{x \to \infty} -\frac{1}{x} = 0$ $\lim_{x \to \infty} \frac{1}{x} = 0$ so $\lim_{x \to \infty} \frac{\sin(x)}{x} = 0$.

c. In Matlab, f(x) = $\frac{\sin(x)}{x}$ on the interval [-10,10] looks like

discontinuity: f(0) not defined.



Without calculation:

On the sketch above, label point(s) of discontinuity: Point(s) of discontinuity of f(x): x = 0

On the sketch above, label the critical points of f(x): Critical points of f(x): -7.5, -5, 5, 7.5 and 0

On the sketch above, label the inflection points of f(x): Inflection points of f(x): -6, -2, 2, 6

On the sketch above, label all regions where $f'(x) > 0$? Answer: (-10, -7.5) (-5, 0) (5, 7.5)

3. Compute the first **derivative** for each of the following functions (show your work):

a $f(x) = x^5 + 5^x + 5^5 = x^5 + e^{x\ln(5)} + 5^5$

$f'(x) =$ $5x^4 + \ln(5)e^{x\ln(5)}$

b. $f(x) = \dfrac{\sin(x)}{x^3+1}$ $\dfrac{(x^3+1)\cos(x) - 3x^2\sin(x)}{(x^3+1)^2}$

$f'(x) =$

Computation:

c. $f(x) = \sqrt{x^3 + \cos^2(x)}$ $\frac{1}{2}\left(x^3 + \cos^2(x)\right)^{-1/2} \cdot \left(3x^2 + 2\cos(x)\cdot-\sin(x)\right)$

$f'(x) =$

Computation

d. $f(x) = \ln(\sin(x)) + \sin^{-1}(2x)$ $\dfrac{1}{\sin(x)} \cdot \cos(x) + \dfrac{1}{\sqrt{1-(2x)^2}}\ 2$

$f'(x) =$

Computation: $y = \sin^{-1}(x)$ $\sin(y) = x$ $\cos(y)\dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = \dfrac{1}{\cos(y)} = \dfrac{1}{\sqrt{1-x^2}}$

4. Find the equation of the tangent line to the curve $x^3 + y^2 + y^4 = 10$ at the point (2,1).
Hint: Use implicit differentiation.

$3x^2 + 2y\dfrac{dy}{dx} + 4y^3\dfrac{dy}{dx} = 0$

$12 + 2\dfrac{dy}{dx} + 4\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{12}{6} = -2$

$y - 1 = -2(x-2)$

$y - 1 = -2(x - 2)$

$(y = -2x + 5)$

5. An oil tanker in the Atlantic Ocean has sprung a leak, creating a circular oil slick. If the area of the oil slick is increasing at a rate of $3m^2$ per minute, how fast is the radius of the oil slick increasing when the radius is 10m.. ANSWER:______________________

$A = \pi r^2$

$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$

$3 = 2\pi \cdot 10 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{20\pi}$ m/s

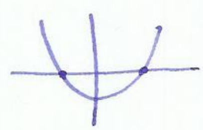
6. Let $f(x) = x^3 - 18x + 5$

a. Where is the function increasing? Answer: $(-\infty, -\sqrt{6})$ $(\sqrt{6}, \infty)$

Show your work:

$f'(x) = 3x^2 - 18$

$f'(x) = 0 : \quad x = \pm\sqrt{6}$



b. What are the inflection point(s)? Answer 0.

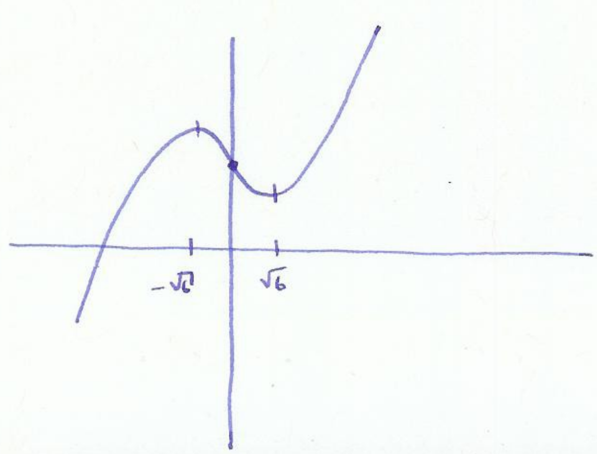
Show your work.

$f''(x) = 6x$

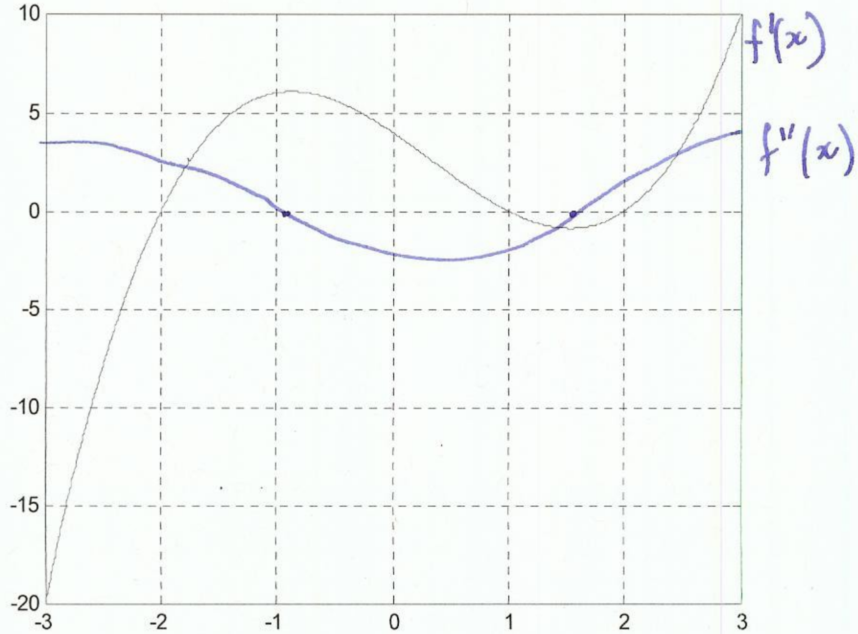
c. Where is the function concave up? Answer: $x > 0$ $(0, \infty)$

Explain: $f''(x) > 0$

d. Draw a graph of the function.



7. The graph of the **derivative of f(x)** is



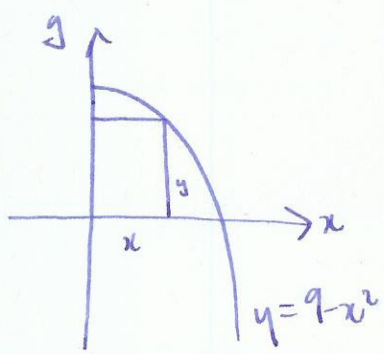
a. Where is the **function f(x)** decreasing?. Answer: $(-3,-2)$ $(1,2)$
Explain:

$f'(x) < 0 \Leftrightarrow$ decreasing

b. Where is the **function f(x)** concave up? Answer: $(-3,-1)$ $(1.5,3)$
Explain:

$f''(x) > 0 \Leftrightarrow$ concave up

8. Find the maximum area of a rectangle inscribed in the region bounded by the graph of $y = 9 - x^2$ and the axes, in the first quadrant. Hint: Sketch this function in the first quadrant. Assume the base of the rectangle is on the x-axis and starts at (0,0). Answer: $6\sqrt{3}$



$A = xy = x(9-x^2) = 9x - x^3$

$\frac{dA}{dx} = 9 - 3x^2$ $x = \sqrt{3}, y = 6$ $A = 6\sqrt{3}$

9 Compute the following antiderivatives (show your work):

a. $\int\left(\frac{1}{\sqrt{x}} + \frac{2}{x} + 1 \right)dx =$ $2x^{1/2} + 2\ln|x| + x + C$

Computation:

b. $\int\left(\cos(2x) + \frac{6}{x\sqrt{x}} \right)dx =$ $\frac{1}{2}\sin(2x) + 6\frac{x^{-1/2}}{-1/2} + C$

Computation:

c. $\int(x + \sec^2(x))\, dx =$ $\frac{1}{2}x^2 + \tan(x) + C$

Computation:

d. $\int\left(x\left(x^2+3\right)^{25} \right)dx =$ ______

Computation:

$u = x^2+3 \quad \frac{du}{dx} = 2x$

$\int x\, u^{25} \frac{1}{2x}\, du = \int \frac{1}{2}u^{25}\, du = \frac{u^{26}}{2 \times 26} + C = \frac{(x^2+3)^{26}}{52} + C$

10. a. Carefully state the First Fundamental Theorem of Calculus

$\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$

$f(x)$ defined on the closed interval $[a,b]$, $F(x)$ also defined on $[a,b]$, differentiable, with $F'(x) = f(x)$.

b Calculate the area under the curve f(x) = sin(x) on the interval [0,π].

Answer: 2

Computation:

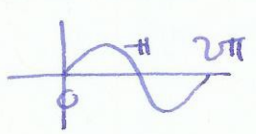
$$\int_0^\pi \sin(x)\,dx = \left[-\cos(x)\right]_0^\pi$$

$$= -\cos(\pi) + \cos(0) = -(-1) + 1 = 2.$$

c. Calculate the area under the curve f(x) = |sin(x)| on the interval [0,2π].

Answer: 4

Computation:



$$\int_0^\pi \sin(x)\,dx - \int_\pi^{2\pi} \sin(x)\,dx$$

11. If F(x) = $\int_0^x \sqrt{1-t^2}\,dt$, then $F'(x) =$ $\sqrt{1-x^2}$