Math 233 Calculus 3 Spring 12 Midterm 3b

Name: Solutions.

- Do any 8 of the following 10 questions.
- You may use a calculator, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 3	
Overall	

(1) (10 points) Use the chain rule to find $\frac{\partial f}{\partial x}$ if

$$f(s,t) = te^{st} \text{ and } s = xy^2, t = 3x + y.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x}$$

$$= t^2 e^{st} y^2 + \left(e^{st} + ste^{st}\right) 3$$

$$= (3x+y)^2 e^{xy^2(3x+y)} y^2 + \left(e^{xy^2(3x+y)}\right)\left(1 + xy^2(3x+y)\right) \cdot 3 .$$

(2) (10 points) Find the critical points of $f(x, y) = x^3 + y^3 - 3xy$ and use the second derivative test to classify them.

$$\left.\begin{array}{l} f_x = 3x^2 - 3y = 0 \\ f_y = 3y^2 - 3x = 0 \end{array}\right\} \quad y = x^2 = y^4 \quad y(y^3-1) = 0 \Rightarrow y = 0, 1$$

$(0,0), (1,1)$

$$\left.\begin{array}{l} f_{xx} = 6x \\ f_{xy} = -3 \\ f_{yy} = 6y \end{array}\right\} \quad D = 36xy - 9$$

$D(0,0) = -9$ saddle

$D(1,1) = 25$ $f_{xx} > 0$ local min

(3) (10 points) Use Lagrange multipliers to find the maximum and minimum values of $4x - 3y$ on the circle $x^2 + y^2 = 9$.

$f(x)$ $\quad g(x)$

$\nabla f = \langle 4, -3 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$:

$\left.\begin{array}{l} 4 = \lambda 2x \\ -3 = \lambda 2y \end{array}\right\} -\frac{4}{3} = \frac{x}{y} \leftrightarrow y = -\frac{3}{4}x$

$g(x) = 9$

$x^2 + y^2 = 9$

$x^2 + \frac{9}{16}x^2 = 9$

$25 \cancel{10}x^2 = 9 \cdot 16$

$\cancel{x = \pm \frac{12}{40}} \Rightarrow \left(\frac{12}{5}, -\frac{9}{5}\right), \left(-\frac{12}{5}, \frac{9}{5}\right).$

$x = \pm \frac{12}{5}$

$f\left(\frac{12}{5}, -\frac{9}{5}\right) = 15$ max

$f\left(-\frac{12}{5}, \frac{9}{5}\right) = -15$ min

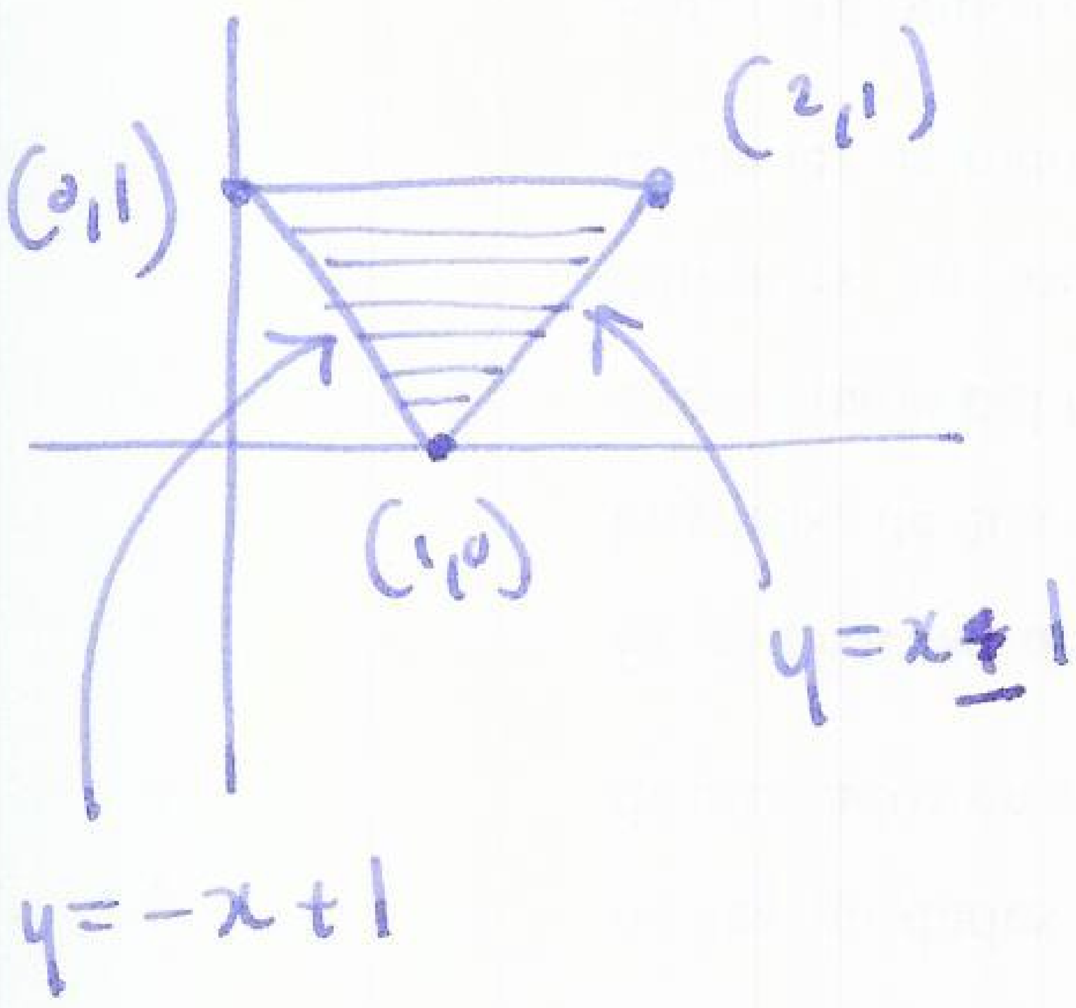
(4) Evaluate $\int_{-1}^{1}\int_{0}^{3} e^{2x+y}\, dxdy.$ $= \int_{-1}^{1} e^{y} \int_{0}^{3} e^{2x}\, dx\, dy$

$\int_0^3 e^{2x}\, dx = \left[\frac{1}{2}e^{2x}\right]_0^3 = \frac{1}{2}(e^6 - 1)$

$\left(\frac{1}{2}(e^6-1)\right)\int_{-1}^{1} e^y\, dy = \frac{1}{2}(e^6-1)\left[e^y\right]_{-1}^{1}$

$= \frac{1}{2}(e^6-1)(e-e^{-1})$.

(5) (10 points) Write down the limits for an integral over the region consisting of the triangle in the xy-plane with vertices $(1,0), (0,1)$ and $(2,1)$.



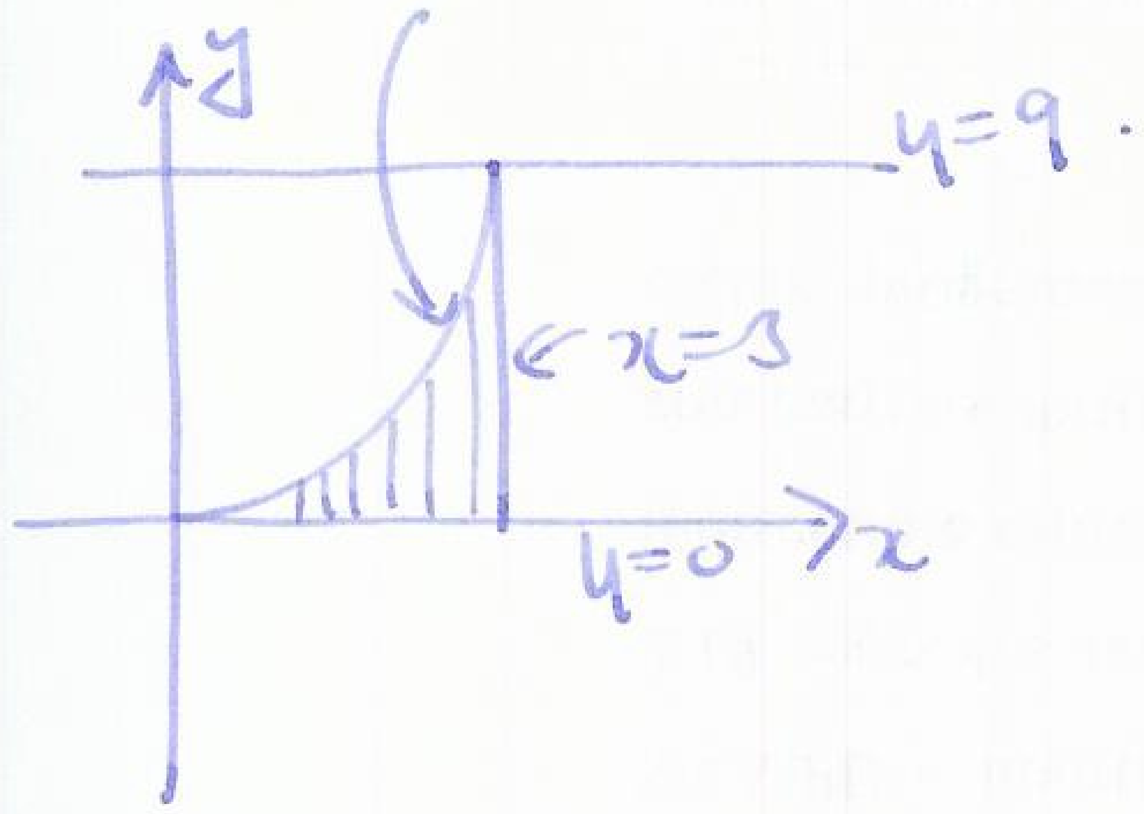
$$\int_0^1 \int_{-y+1}^{y+1} f(x,y)\, dx\, dy$$

$x^2 = y$

$x = \sqrt{y}$

(6) (10 points) Evaluate the following integral by changing the order of integration.

$$\int_0^9 \int_{\sqrt{y}}^3 \sin(x^3)\, dx dy$$



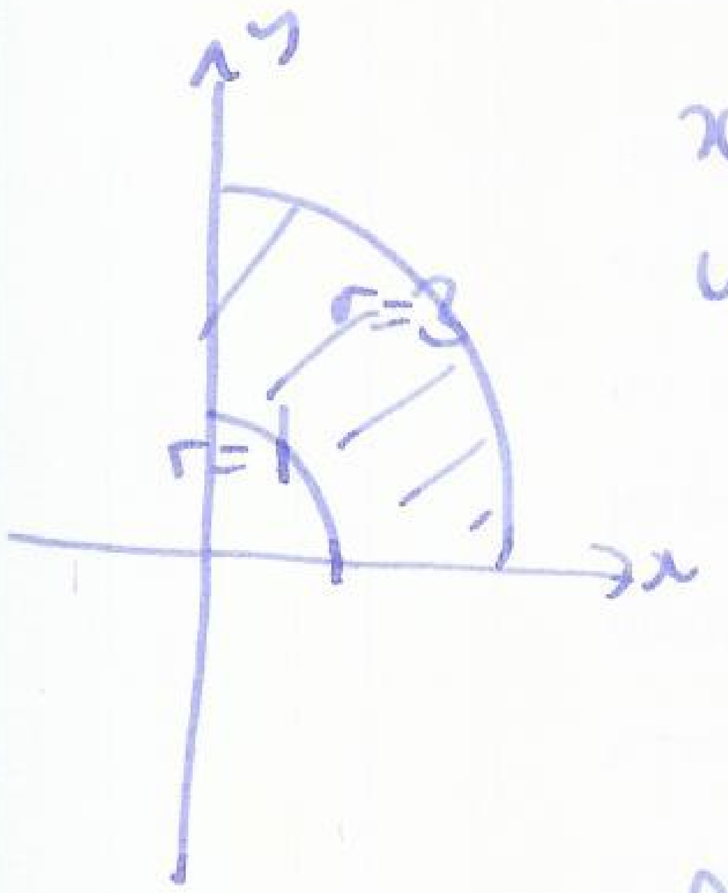
$$\int_0^3 \int_0^{x^2} \sin(x^3)\, dy\, dx$$

$$\left[y \sin(x^3) \right]_0^{x^2} = x^2 \sin(x^3).$$

$$\int_0^3 x^2 \sin(x^3)\, dx = \left[-\frac{1}{3} \cos(x^3) \right]_0^3.$$

$$= -\frac{1}{3} \cos(27) + \frac{1}{3}.$$

(7) (10 points) Integrate the function $f(x, y) = \dfrac{x}{x^2 + y^2}$ in the region between the circle of radius 1 and the circle of radius 3, which lies in the first quadrant, i.e. $x \geq 0, y \geq 0$. (Hint: use polar coordinates.)



$x = r\cos\theta$

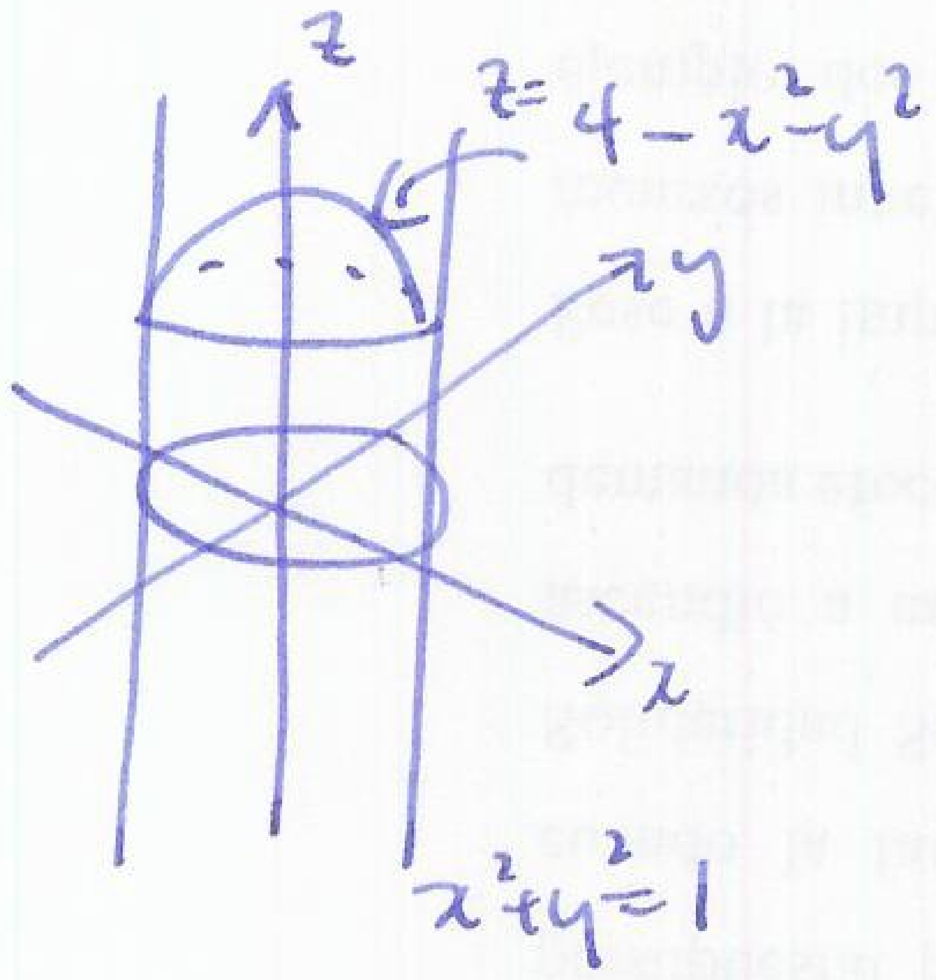
$y = r\sin\theta$

$$\int_0^{\pi/2} \int_1^3 \frac{r\cos\theta}{r^2} \, r \, dr \, d\theta$$

$$\int_1^3 \cos\theta \, dr = 2\cos\theta$$

$$\int_0^{\pi/2} 2\cos\theta \, d\theta = \Big[+2\sin\theta\Big]_0^{\pi/2} = \cancel{2 - 2\sin} \; +2\sin\left(\tfrac{\pi}{2}\right) - 2\sin(0) = +2$$

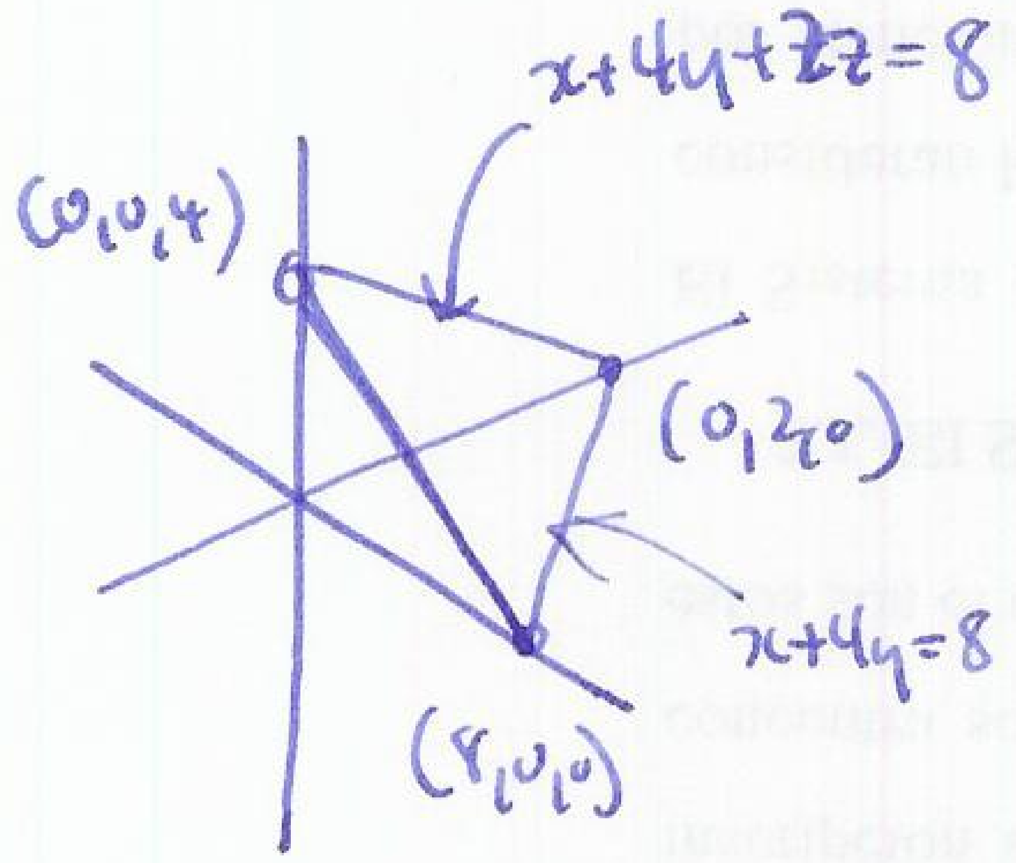
(8) (10 points) Write down limits for an integral over the region inside the cylinder $x^2 + y^2 = 1$, beneath the surface $z = 4 - x^2 - y^2$, and above the xy=plane. You may use any coordinate system.



use ~~polar~~ cylindrical:

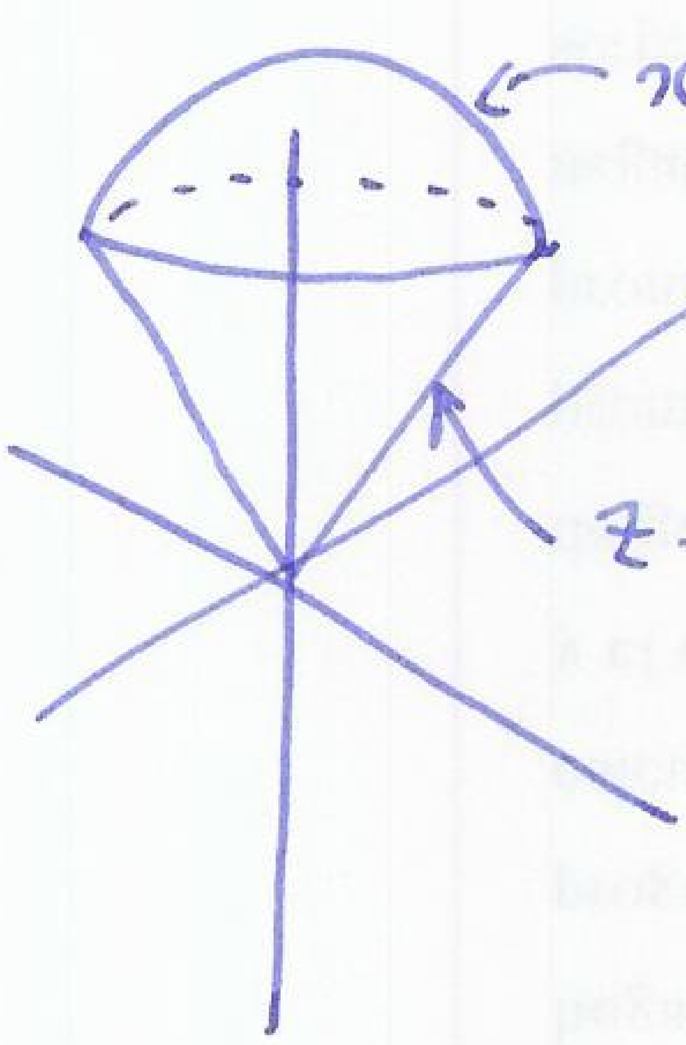
$$\int_0^{2\pi} \int_0^1 \int_0^{4-r^2} f(x,y,z) \; r\,dz\,dr\,d\theta$$

(9) (10 points) Write down limits for a 3-dimensional integral over the region in the positive octant (i.e. $x \geq 0, y \geq 0, z \geq 0$), below the plane $x + 4y + 2z = 8$.



$$\int_0^8 \int_0^{2-\frac{x}{4}} \int_0^{4-\frac{x}{2}-2y} f(x,y,z)\,dz\,dy\,dx$$

(10) (10 points) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$, which lies above the cone $z = 2\sqrt{x^2 + y^2}$.



$\leftarrow x^2+y^2+z^2 = 4 \Leftrightarrow \rho^2 = 4 \Leftrightarrow \rho = 2$

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$

$z = 2\sqrt{x^2+y^2} \Leftrightarrow \rho\cos\phi = 2\rho\sin\phi \Leftrightarrow \tan\phi = \frac{1}{2} \Leftrightarrow \phi = \cancel{\frac{\pi}{6}} \approx 0.46$

$$\int_0^{\cancel{\pi/6}\ 0.46} \int_0^2 \underbrace{\int_0^{2\pi} 1\ \rho^2 \sin\phi \ d\theta}_{= 2\pi \rho^2 \sin\phi} \ d\rho \ d\phi$$

$$\int_0^2 \rho^2 \, d\rho = \left[\frac{1}{3}\rho^3\right]_0^2 = \frac{8}{3}. \qquad \int_0^{\cancel{\pi/6}\ 0.46} \sin\phi \, d\phi = \Big[-\cos(\phi)\Big]_0^{\cancel{\pi/6}\ 0.46} = 1 + \cos(0.46)$$

so volume $= \frac{16\pi}{3} \cdot 1.46$