Math 233 Calculus 3 Spring 12 Midterm 3a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 3	
Overall	

(1) (10 points) Use the chain rule to find $\frac{\partial f}{\partial x}$ if

$$f(s,t) = te^{st} \text{ and } s = x^2y, t = x + 3y.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x}$$

$$= t^2 e^{st} \cdot 2xy + \left(se^{st} + e^{st}\right) 1$$

$$= (x+3y)^2 e^{x^2y(x+3y)} 2x^2y(x+3y) + (x^2y)e^{x^2y(x+3y)} + e^{x^2y(x+3y)}$$

(2) (10 points) Find the critical points of $f(x, y) = x^3 + y^3 - 2xy$ and use the second derivative test to classify them.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 2y = 0 \\ \frac{\partial f}{\partial y} &= 3y^2 - 2x = 0 \end{aligned}\right\} \quad \begin{aligned} &y = \frac{3}{2}x^2 = \frac{3}{2}\left(\frac{3}{2}y^2\right)^2 = \frac{27}{8}y^4 \\ &y\left(1 - \frac{27}{8}y^3\right) = 0 \quad y = 0, \frac{2}{3} \end{aligned}$$

$$\text{so: } (0,0), \left(\frac{2}{3}, \frac{2}{3}\right)$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 6x \\ f_{xy} &= -2 \\ f_{yy} &= 6y \end{aligned}\right\} \quad \begin{aligned} &D = 36xy - 4 \\ &D(0,0) = -4 \quad \text{saddle} \\ &D\left(\frac{2}{3}, \frac{2}{3}\right) = 16 \quad f_{xx}\left(\frac{2}{3}, \frac{2}{3}\right) > 0 \Rightarrow \text{local min} \end{aligned}$$

(3) (10 points) Use Lagrange multipliers to find the maximum and minimum values of $3x - 4y$ on the circle $x^2 + y^2 = 4$.

$f(x)$ $\quad$ $g(x) = 4$

$\nabla f = \langle 3, -4 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$: $\left.\begin{array}{l} 3 = \lambda 2x \\ -4 = \lambda 2y \end{array}\right\}$ $-\frac{3}{4} = \frac{x}{y}$ $\quad$ $y = \frac{-4x}{3}$.

$x^2 + y^2 = 4$

$x^2 + \frac{16x^2}{9} = 4$.

$9x^2 + 16x^2 = 36$

$25x^2 = 36$ $\quad$ $x = \pm \frac{6}{5}$ $\quad$ $\left(\frac{6}{5}, -\frac{8}{5}\right)$

$\left(-\frac{6}{5}, \frac{8}{5}\right)$.

$f\left(\frac{6}{5}, -\frac{8}{5}\right) = 10$ max

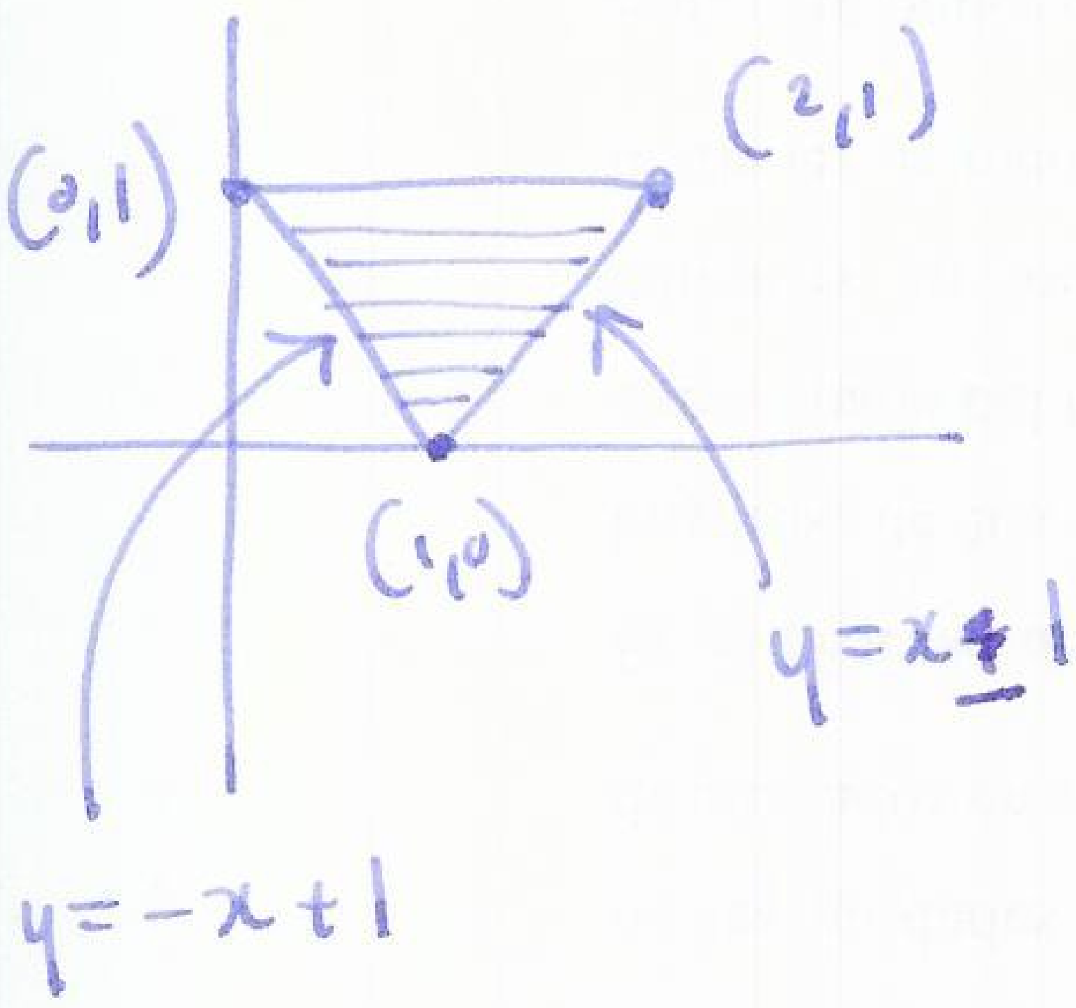
$f\left(-\frac{6}{5}, \frac{8}{5}\right) = -10$ min

(4) Evaluate $\int_{-1}^{1}\int_{0}^{2} e^{x+2y}\,dxdy.$

$$\left[e^{x+2y}\right]_0^2 = e^{2+2y} - e^{2y} = e^{2y}(e^2-1)$$

$$(e^2-1)\int_{-1}^{1} e^{2y}\,dy = (e^2-1)\left[\tfrac{1}{2}e^{2y}\right]_{-1}^{1} = \tfrac{1}{2}(e^2-1)(e^2-e^{-2}).$$

(5) (10 points) Write down the limits for an integral over the region consisting of the triangle in the xy-plane with vertices $(1,0)$, $(0,1)$ and $(2,1)$.



$$\int_0^1 \int_{-y+1}^{y+1} f(x,y)\, dx\, dy$$

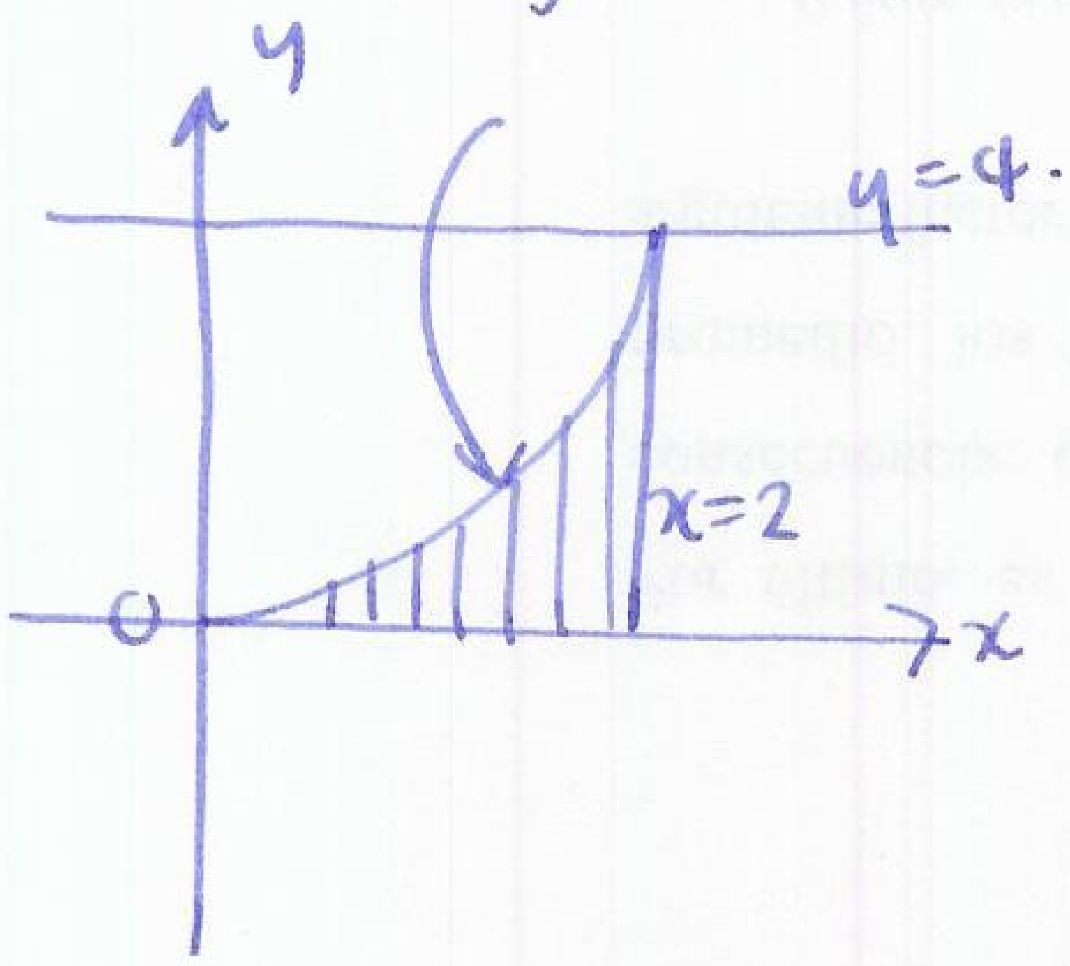
$x^2 = y$

$\Updownarrow$

$x = \sqrt{y}$

(6) (10 points) Evaluate the following integral by changing the order of integration.

$$\int_0^4 \int_{\sqrt{y}}^2 \sin(x^3)\, dx dy$$



$$\int_0^2 \int_0^{x^2} \sin(x^3)\, dy\, dx$$

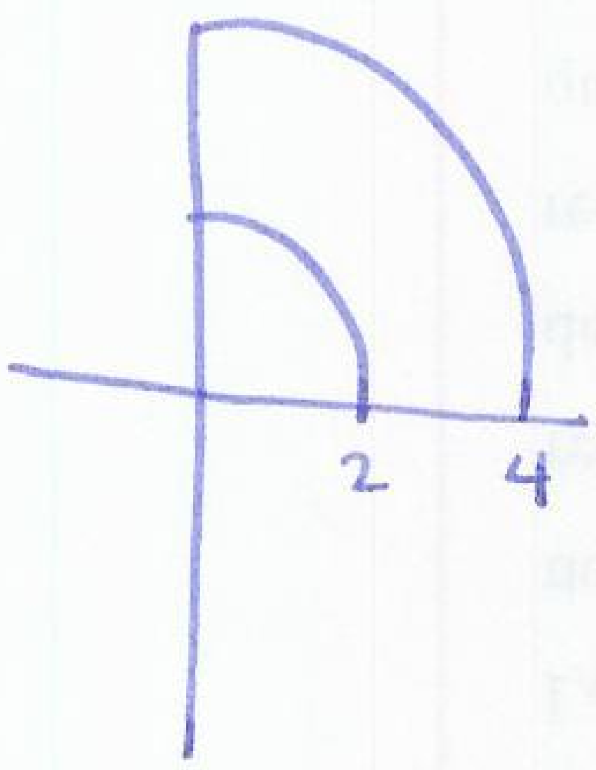
$$\left[\sin(x^3)\, y\right]_0^{x^2} = x^2 \sin(x^3).$$

$$\int_0^2 x^2 \sin(x^3)\, dx = \left[-\cos(x^3)\right]_0^2$$

$$= 1 - \cos(8).$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

(7) (10 points) Integrate the function $f(x, y) = \dfrac{x}{x^2 + y^2}$ in the region between the circle of radius 2 and the circle of radius 4, which lies in the first quadrant, i.e. $x \geq 0, y \geq 0$. (Hint: use polar coordinates.)

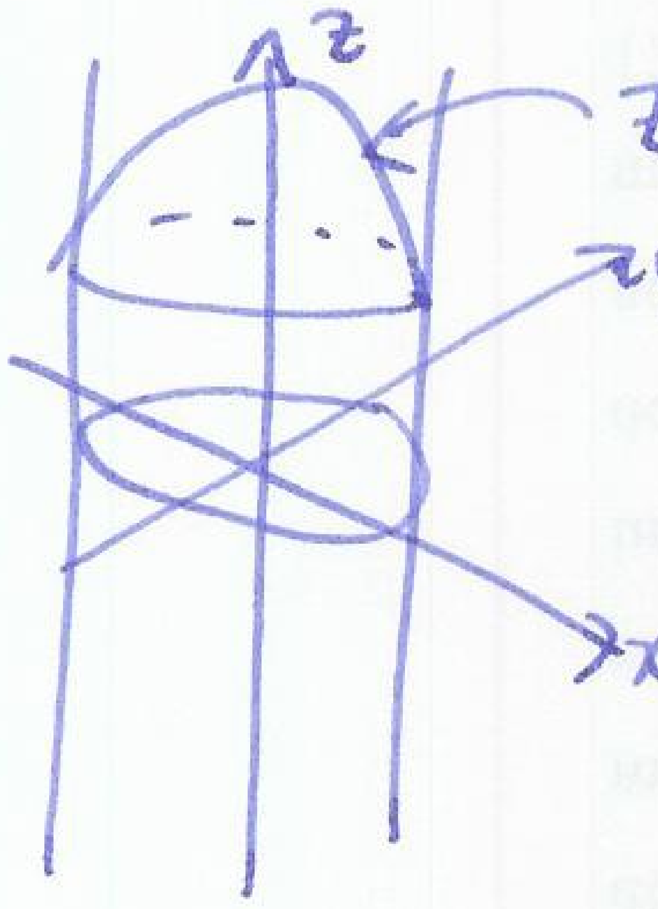


$$\int_0^{\pi/2} \underbrace{\int_2^4 \frac{r\cos\theta}{r^2}\, r\, dr}\, d\theta$$

$$\cos\theta \int_2^4 dr = 2\cos\theta$$

$$\int_0^{\pi/2} 2\cos\theta\, d\theta = \Big[2\sin\theta\Big]_0^{\pi/2} = 2\sin\left(\tfrac{\pi}{2}\right) = 2.$$

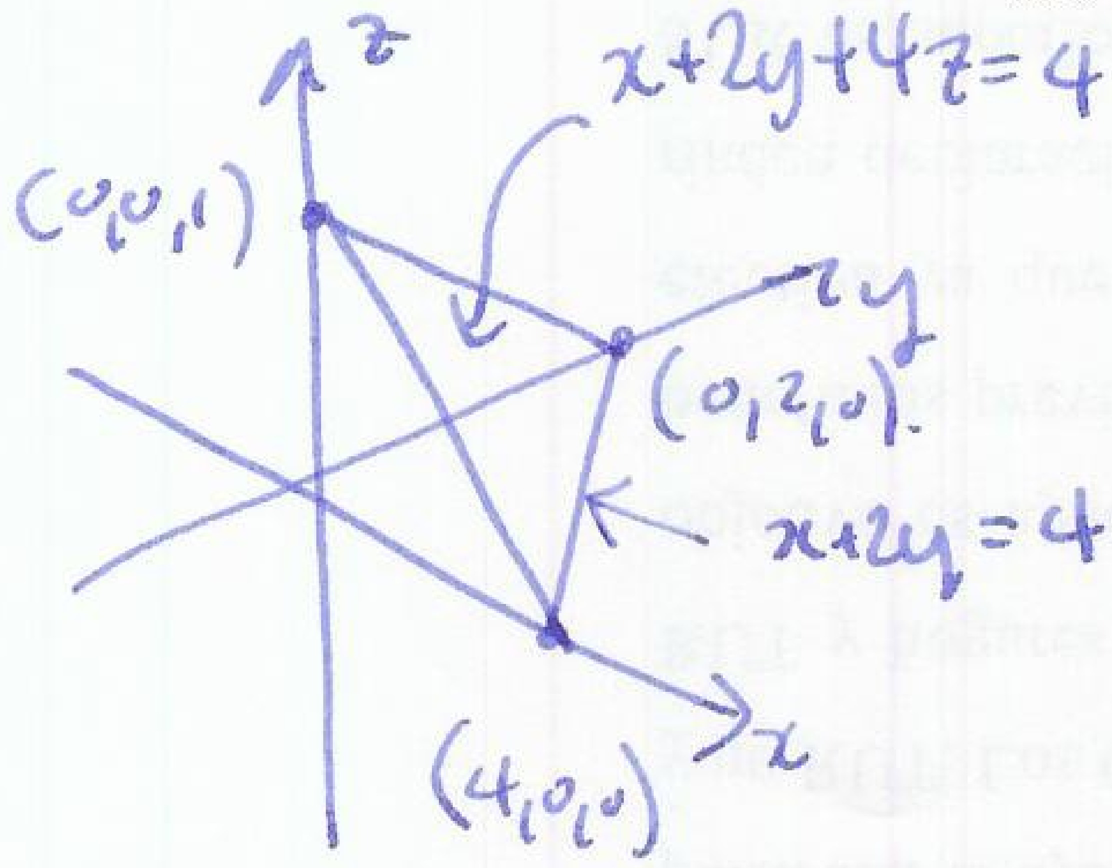
(8) (10 points) Write down limits for an integral over the region inside the cylinder $x^2 + y^2 = 1$, beneath the surface $z = 8 - x^2 - y^2$, and above the xy=plane. You may use any coordinate system.



$z = 8 - x^2 - y^2$

polars: $\displaystyle\int_0^{2\pi}\int_0^{1}\int_0^{8-r^2} f(x,y,z)\, r\, dz\, dr\, d\theta$

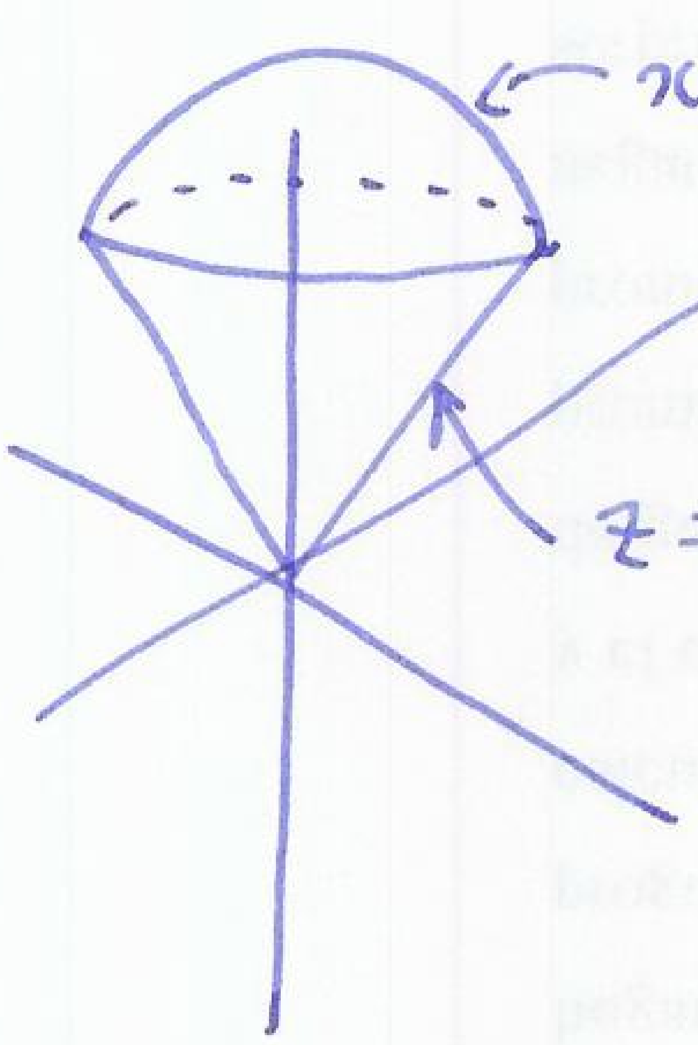
(9) (10 points) Write down limits for a 3-dimensional integral over the region in the positive octant (i.e. $x \geq 0, y \geq 0, z \geq 0$), below the plane $x+2y+4z=4$.



$$\int_0^4 \int_0^{2-\frac{x}{2}} \int_0^{1-\frac{y}{2}-\frac{x}{4}} f(x,y,z)\ dz\, dy\, dx$$

(10) (10 points) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$, which lies above the cone $z = 2\sqrt{x^2 + y^2}$.

$x = \rho \cos\theta \sin\phi$
$y = \rho \sin\theta \sin\phi$
$z = \rho \cos\phi$



$\leftarrow x^2+y^2+z^2 = 4 \Leftrightarrow \rho^2 = 4 \Leftrightarrow \rho = 2$

$z = 2\sqrt{x^2+y^2} \Leftrightarrow \rho\cos\phi = 2\rho\sin\phi \Leftrightarrow \tan\phi = \frac{1}{2} \Leftrightarrow \phi = \cancel{\frac{\pi}{6}} \approx 0.46$

$$\int_0^{\cancel{\pi/6}\ 0.46} \int_0^2 \underbrace{\int_0^{2\pi} 1\, \rho^2 \sin\phi \, d\theta}_{=\ 2\pi \rho^2 \sin\phi} \, d\rho \, d\phi$$

$$\int_0^2 \rho^2 \, d\rho = \left[\frac{1}{3}\rho^3\right]_0^2 = \frac{8}{3}. \qquad \int_0^{\cancel{\pi/6}\ 0.46} \sin\phi \, d\phi = \left[-\cos(\phi)\right]_0^{\cancel{\pi/6}\ 0.46} = 1 + \cos(0.46)$$

so volume $= \frac{16\pi}{3} \cdot 1.46$