Sample midterm 3 Solutions

Q1 $\mathbb{R}^2 \xrightarrow[g]{} \mathbb{R}^2 \xrightarrow[f]{} \mathbb{R}^2$ chain rule: $D(f\circ g) = Df(g)\,Dg$

(x,y) (u,v)

$g(x,y) = (x+y, x-y)$ $Dg = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$f(u,v) = (u^2+v^2, uv)$ $Df = \begin{bmatrix} 2u & 2v \\ v & u \end{bmatrix}$

$$Df(g)\,Dg = \begin{bmatrix} 2x+2y & 2x-2y \\ x-y & x+y \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4x & 4y \\ 2x & 2y \end{bmatrix}.$$

Q2 a) $f(x,y) = x^3 - xy + y^3$

$f_x = 3x^2 - y = 0$, $f_y = -x + 3y^2 = 0$ $\Big\}$ $y = 3x^2$, $x = 3y^2$ $\quad x = 27x^4 \quad x(1-27x^3) = 0$

$x = 0 \Rightarrow y = 0 \qquad x = \frac{1}{3} \Rightarrow y = \frac{1}{3}$

$f_{xx} = 6x$

$f_{xy} = -1$ $\qquad D = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 1$

$f_{yy} = 6y$

$D(0,0) < 0 \Rightarrow$ saddle

$D(\frac{1}{3},\frac{1}{3}) > 0 \quad f_{xx} > 0 \Rightarrow$ local min.

b) $f(x,y) = e^x - xe^y$

$f_x = e^x - e^y = 0 \Rightarrow y = 0$ $\qquad$ one critical point at $(0,0)$

$f_y = -xe^y = 0 \Rightarrow x = 0$

$f_{xx} = e^x$

$f_{xy} = -e^y$ $\qquad D = -xe^{x+y} - e^{2y}$

$f_{yy} = -xe^y$ $\qquad D(0,0) = -1 \Rightarrow$ saddle.

c) $f(x,y) = x\ln(x+y)$

$$\left.\begin{aligned} f_x &= \ln(x+y) + \frac{x}{x+y} = 0 \\ f_y &= \frac{x}{x+y} = 0 \end{aligned}\right\} \Rightarrow x = 0 \qquad \Rightarrow y = 1$$

$(0,1)$ critical point.

$$f_{xx} = \frac{1}{x+y} + \frac{(x+y) - x}{(x+y)^2}$$

$$f_{xy} = \frac{(x+y)-x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$f_{yy} = \cancel{\frac{1}{(x+y)}} \; \frac{-x}{(x+y)^2}$$

$$D(0,1) = (1+1)\cdot(0) - (1)^2 = -1$$

$\Rightarrow$ saddle.

Q3 max $f(x,y) = x^2y + x + y$ subject to $g(x,y) = xy = 4$.

$\nabla f = \langle 2xy+1,\; x^2+1 \rangle \qquad \nabla g = \langle y, x \rangle$

$\nabla f = \lambda \nabla g$:

$$\left.\begin{aligned} 2xy + 1 &= \lambda y \\ x^2 + 1 &= \lambda x \\ xy &= 4 \end{aligned}\right\} \qquad \left.\frac{2xy+1}{x^2+1} = \frac{y}{x}\right\} \Rightarrow 2x^2y + x = x^2y + y$$

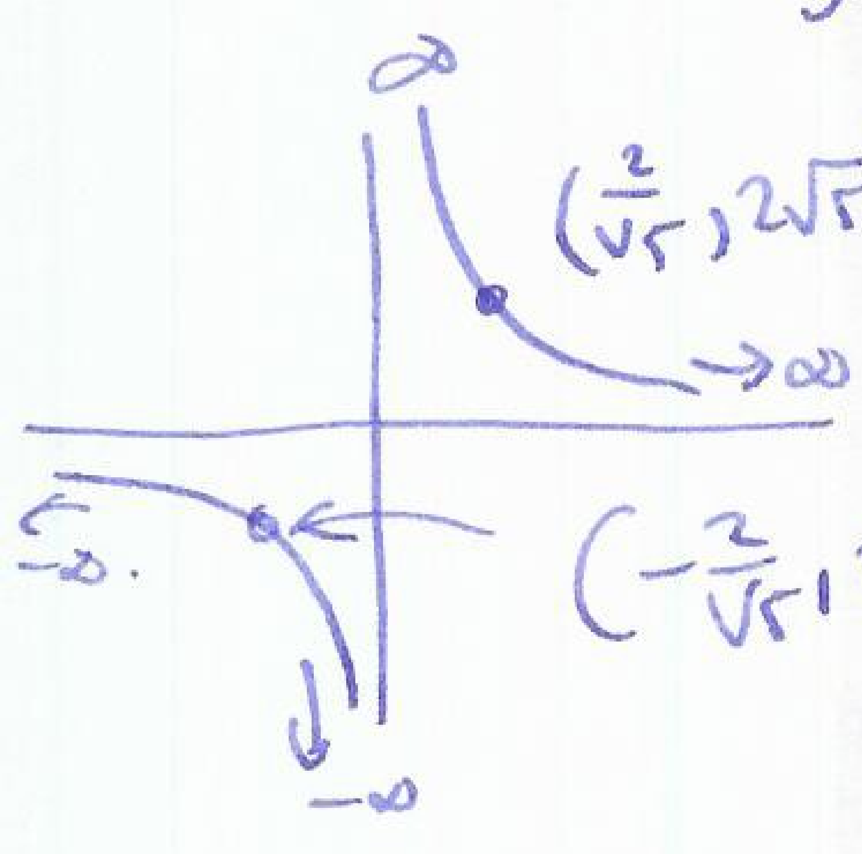
$$x^2y + x = y$$

$xy = 4 \Rightarrow 4x + x = y$

$$5x = y$$

$y = \frac{4}{x} \Rightarrow$

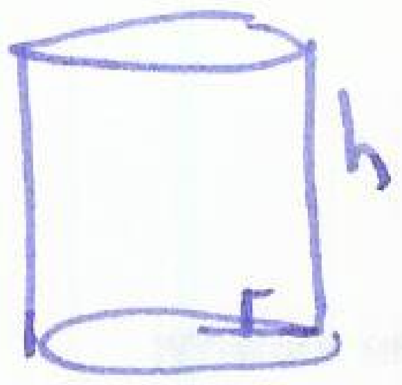
$$5x = \frac{4}{x} \qquad x^2 = \frac{4}{5} \qquad x = \pm\frac{2}{\sqrt{5}},\; y = \pm 2\sqrt{5}$$



$\left(\frac{2}{\sqrt{5}}, 2\sqrt{5}\right) \leftarrow$ min on this branch

$\left(-\frac{2}{\sqrt{5}}, 2\sqrt{5}\right) \leftarrow$ max on lower branch

Q4



$V = \pi r^2 h \leftarrow$ subject to

$A = 2\pi r^2 + hr$ min

$\nabla A = \langle 4\pi r + h,\ r \rangle$

$\nabla V = \langle 2\pi r h,\ \pi r^2 \rangle$

$$\left.\begin{aligned} 4\pi r + h &= \lambda 2\pi r h \\ r &= \lambda \pi r^2 \end{aligned}\right\} 1 = \lambda \pi r$$

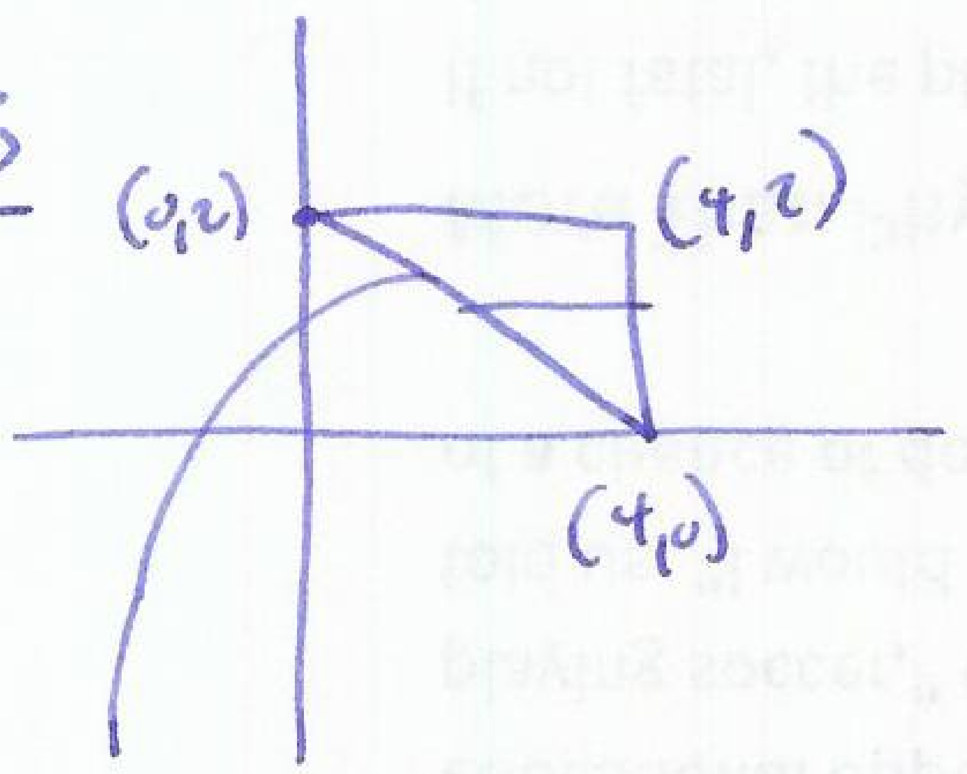
$$\pi r^2 h = V$$

$$\left.\begin{aligned} 4\pi r + h &= 2h \\ \pi r^2 h &= V \end{aligned}\right\} \quad h = 4\pi r$$

$\pi r^2 \cdot 4\pi r = V \qquad r^3 = \frac{V}{4\pi^2} \qquad r = \sqrt[3]{\frac{V}{4\pi^2}}$

$$h = \frac{V}{\pi\left(\frac{V}{4\pi^2}\right)^{2/3}}.$$

Q5



$$\int_0^2 \int_{4-2y}^4 xy \, dx\, dy$$

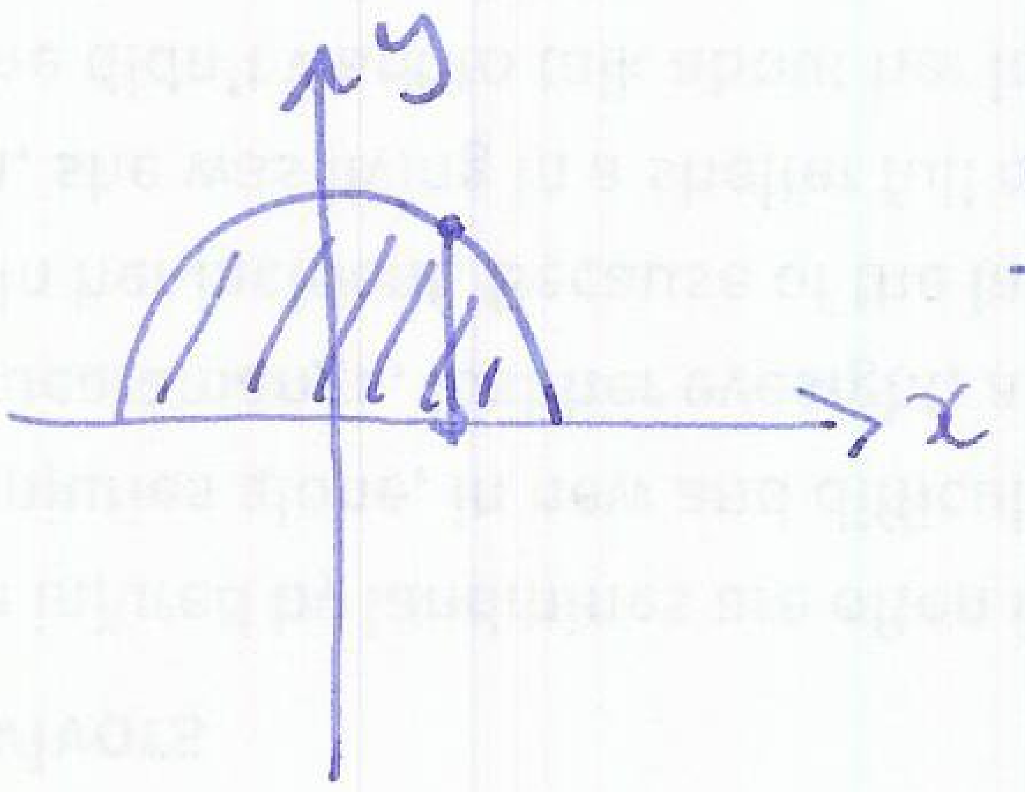
$x + 2y = 4$

$$\left[\frac{1}{2}x^2 y\right]_{4-2y}^4 = 8y - \frac{1}{2}y(4-2y)^2$$

$$= 8y - \frac{1}{2}y(16 - 16y + 4y^2).$$

$$\int_0^2 8y^2 - 2y^3 \, dy = \left[\frac{8}{3}y^3 - \frac{1}{2}y^4\right]_0^2 = \frac{64}{3} - 8.$$

Q6 a)



$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \frac{y}{(1+x^2+y^2)^2} \, dy\, dx$$

$$= \left[\frac{-1/2}{1+x^2+y^2} \right]_0^{\sqrt{1-x^2}} = \frac{-1/2}{2} + \frac{1/2}{1+x^2}$$

$$\int_{-1}^{1} -\frac{1}{4} + \frac{1/2}{1+x^2}\, dx = \left[-\frac{1}{4}x + \frac{1}{2}\tan^{-1}(x) \right]_{-1}^{1}$$

$$= -\frac{1}{2} + \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right) = -\frac{1}{2} + \frac{\pi}{4}$$

b) $$\int_0^{\pi} \int_0^1 \frac{r\sin\theta}{(1+r^2)^2}\, r\,dr\,d\theta$$

$$\int_0^1 \frac{r}{(1+r^2)^2}\, dr \quad \int_0^{\pi} \sin\theta\, d\theta = \left[-\cos\theta \right]_0^{\pi} = -\cos(\pi) + 1 = 2.$$

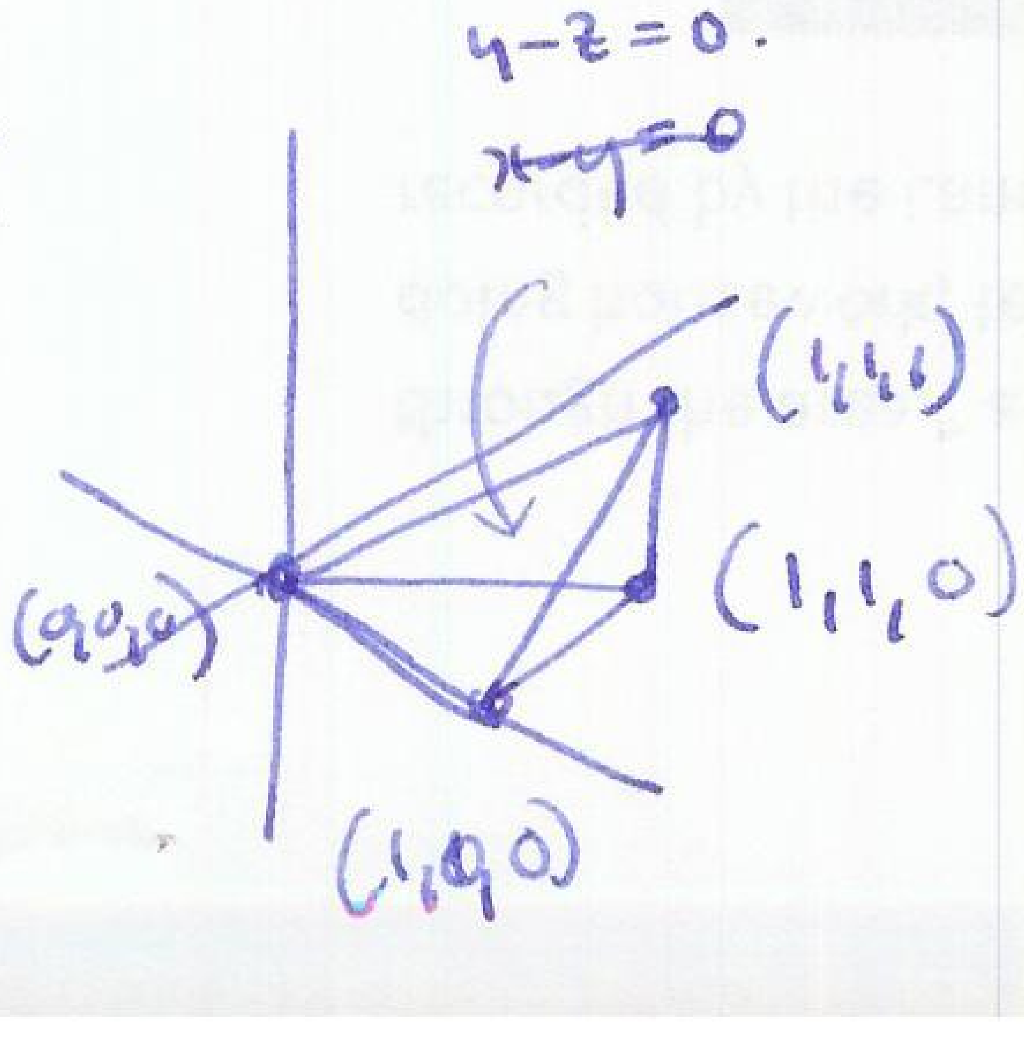
$$\int_0^{\cancel{\pi}1} \frac{r^2}{(1+r^2)^2}\, dr \quad (\text{yuk!}) \quad \text{sub} \quad r = \tan\theta \quad \frac{dr}{d\theta} = \sec^2\theta$$

$$= \int_0^{\pi/4} \frac{\tan^2\theta}{(1+\tan^2\theta)^2} \sec^2\theta\, d\theta = \int_0^{\pi/4} \frac{\tan^2\theta}{\sec^4\theta} \sec^2\theta\, d\theta = \int_0^{\pi/4} \frac{\tan^2\theta}{\sec^2\theta}\, d\theta = \int_0^{\pi/4} \sin^2\theta\, d\theta$$

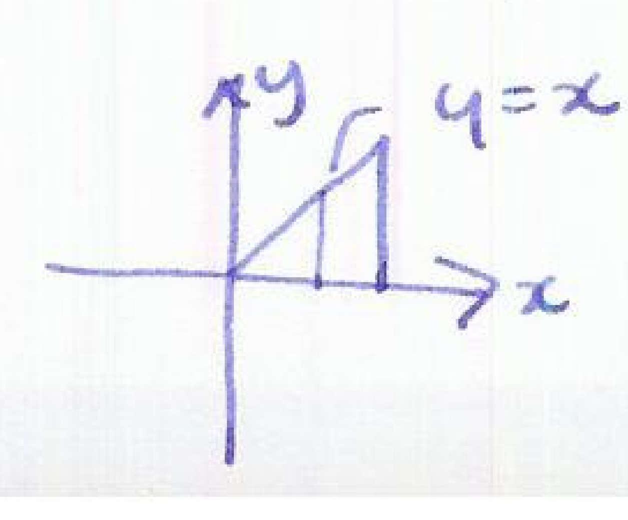
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta \qquad \int_0^{\pi/4} \frac{1}{2} - \frac{1}{2}\cos 2\theta\, d\theta = \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta \right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}.$$

$$= 2\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

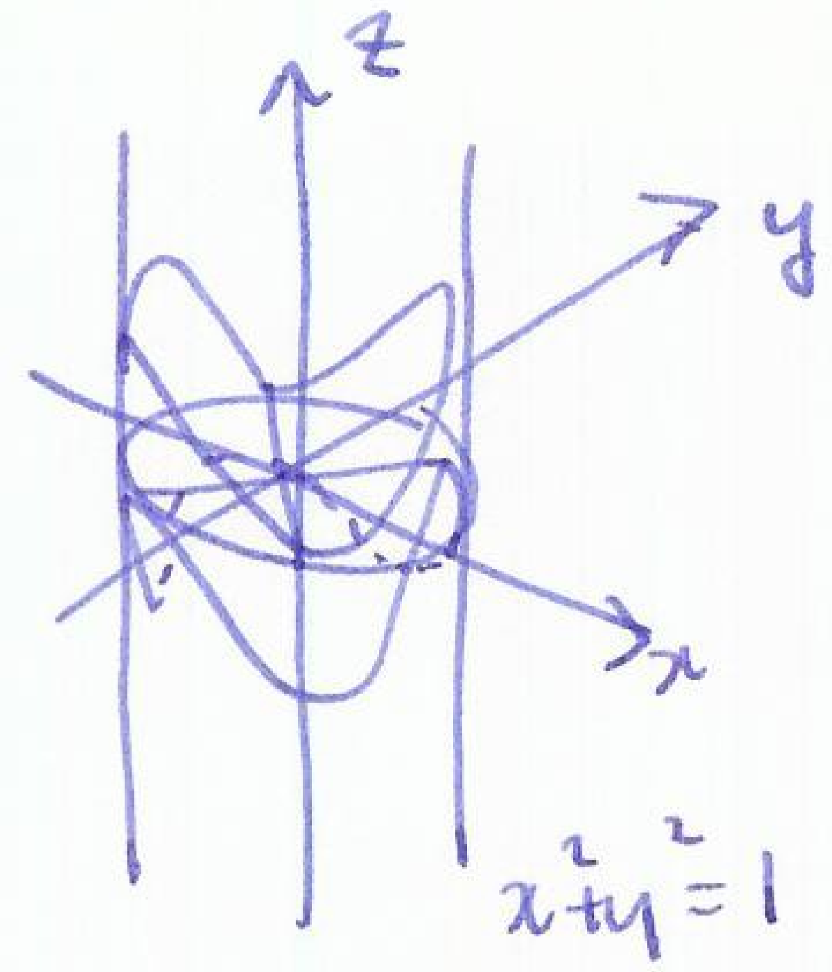
Q7



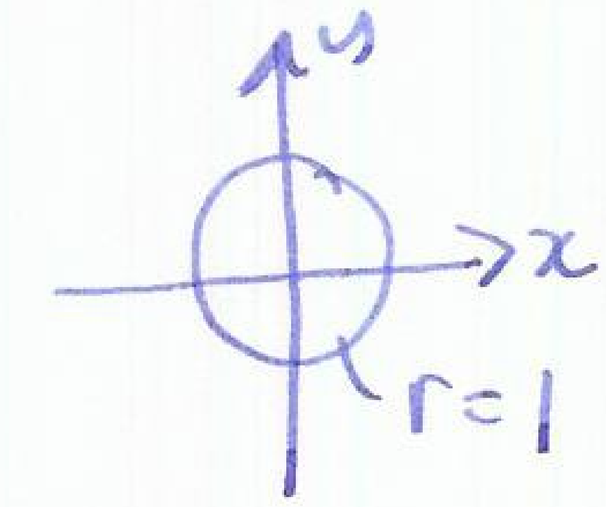
$$\int_0^1 \int_0^x \int_0^y f(x,y,z)\, dz\, dy\, dx$$



Q8



$$\int_0^{2\pi}\int_0^1\int_{\underbrace{-(r\cos\theta-r\sin\theta)^2}}^{(r\cos\theta+r\sin\theta)^2} 1 \quad r\,dz\,dr\,d\theta$$



$$\Big[\, r z \,\Big]_{-r^2(\cos\theta-\sin\theta)^2}^{r^2(\cos\theta+\sin\theta)^2}$$

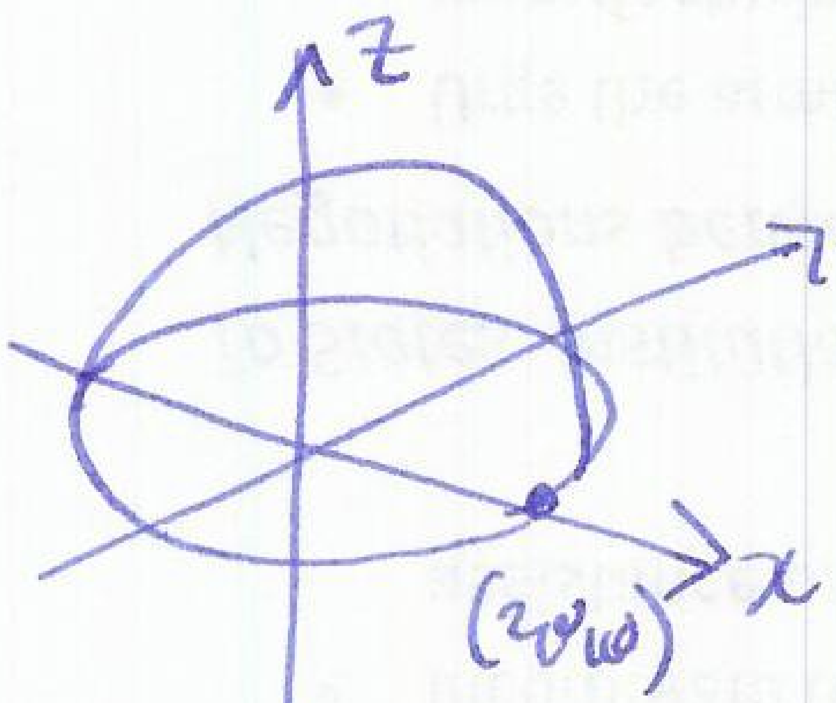
$$= r^3\left(\frac{\cancel{2\sin\theta\cos\theta+2\sin\theta\cos\theta}}{2\cos^2\theta+2\sin^2\theta}\right) = \cancel{2r^3\sin 2\theta}$$

$$= 2r^3.$$

$$\int_0^1 2r^3\,dr = \left[\tfrac{1}{2}r^4\right]_0^1 = \tfrac{1}{2}.$$

$$\int_0^{2\pi}\tfrac{1}{2}\,d\theta = \pi.$$

Q9



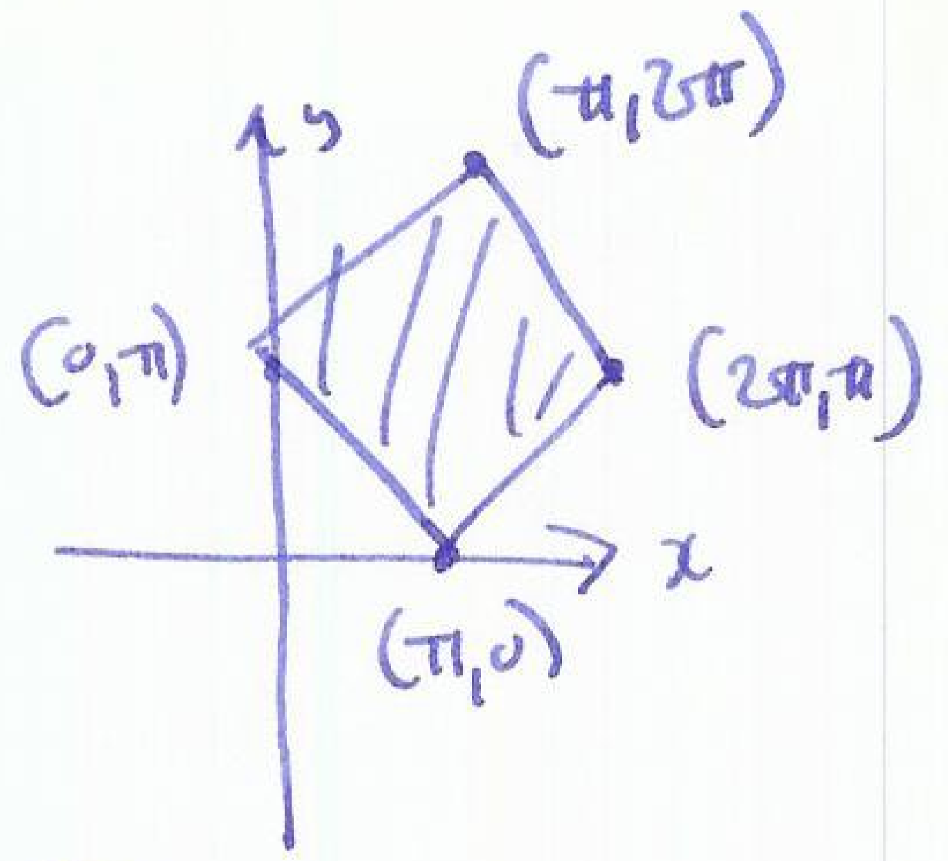
$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^2 e^{-\rho^3}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

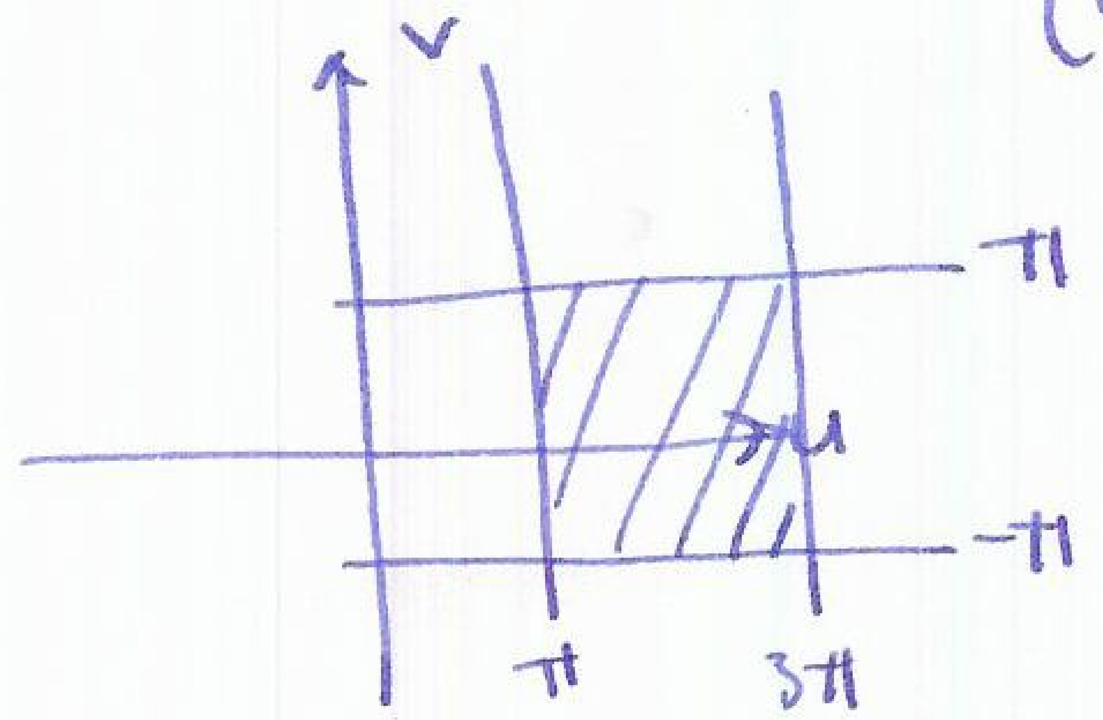
$$\int_0^2 -\rho^2 e^{-\rho^3}\,d\rho = \left[\tfrac{1}{3}e^{-\rho^3}\right]_0^2 = \tfrac{1}{3}(-e^{-8}+1)$$

$$\int_0^{2\pi} d\theta = 2\pi \qquad \int_0^{\pi/2}\sin\phi\,d\phi = \Big[-\cos\phi\Big]_0^{\pi/2} = +1$$

$$= \frac{2\pi}{3}\left(1-e^{-8}\right).$$

Q10





$$(u, v) \mapsto \left(\underbrace{\frac{u+v}{2}}_{x}, \underbrace{\frac{u-v}{2}}_{y}\right)$$

boundaries:

$$x+y=\pi \iff u=\pi$$
$$x+y=3\pi \iff u=3\pi$$
$$y=x+\pi \iff v=-\pi$$
$$y=x-\pi \iff v=\pi$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix}\right| = \frac{1}{2}$$

$$\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} (v\sin(u))^2 \, \frac{1}{2} \, du\,dv$$

$$\int_{-\pi}^{\pi} v^2\,dv = \left[\frac{1}{3}v^3\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^3$$

$$\int_{\pi}^{3\pi} \sin^2 u\,du = \int_{\pi}^{3\pi} \frac{1}{2}+\frac{1}{2}\cos 2u\,du = \pi.$$

$$\text{answer} = \frac{1}{2}\cdot\frac{2}{3}\pi^3\,\pi = \frac{1}{3}\pi^4$$