Math 233 Calculus 3 Spring 12 Final b

Name: Solutions

- Do any 10 of the following 12 questions.
- You may use a calculator, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Midterm 3	
Overall	

(1) (10 points) Let **u** be the vector $\langle 1, -2, 2\rangle$, and let **v** be the vector $\langle 2, -1, 2\rangle$.

(a) Write **v** as the sum of two vectors, one parallel to **u**, and one perpendicular to **u**.

(b) Find the area of the triangle formed by the two vectors **u** and **v**.

a)

$$\frac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|^2}\underline{u} = \frac{2+2+4}{1+4+4}\underline{u} = \frac{8}{9}\langle 1,-2,2\rangle$$

$$\underline{v} - \frac{8}{9}\underline{u} = \langle 2,-1,2\rangle - \frac{8}{9}\langle 1,-2,2\rangle = \left\langle \frac{10}{9}, \frac{7}{9}, \frac{2}{9}\right\rangle$$

$$\underline{v} = \underbrace{\left\langle \frac{8}{9}, \frac{-16}{9}, \frac{16}{9}\right\rangle}_{\text{parallel}} + \underbrace{\left\langle \frac{10}{9}, \frac{7}{9}, \frac{2}{9}\right\rangle}_{\text{perpendicular}}$$

b) $$\underline{u}\times\underline{v} = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 2 & -1 & 2 \end{vmatrix} = \langle -2, 2, 3\rangle$$

$$\text{area} = \frac{1}{2}\|\underline{u}\times\underline{v}\| = \frac{1}{2}\sqrt{4+4+9} = \frac{1}{2}\sqrt{17}$$

(2) (10 points)

(a) Give an example of a parameterized curve in $\mathbb{R}^3$ which has constant speed, but not constant velocity.

(b) A particle starts at the origin at time 0, and has velocity given by $\mathbf{r}'(t) = \langle e^{-3t}, t, 1\rangle$. Where is it at time $t = 2$?

a) $\underline{r}(t) = \langle \cos t, \sin t, 0 \rangle$ check: $\underline{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$

↑ not constant.

$\|\underline{r}'(t)\| = \sin^2 t + \cos^2 t = 1$.

b) $\underline{r}(t) = \langle -\frac{1}{3}e^{-3t}, \frac{1}{2}t^2, t \rangle + \underline{c}$

$\underline{r}(0) = \underline{0} \Rightarrow \langle -\frac{1}{3}, 0, 0 \rangle + \underline{c} = \underline{0}$ $\quad \underline{c} = \langle \frac{1}{3}, 0, 0 \rangle$

at $t=2$: $\underline{r}(2) = \langle -\frac{1}{3}e^{-6}, 2, 2 \rangle + \langle \frac{1}{3}, 0, 0 \rangle$

$= \langle \frac{1}{3}(1-e^{-6}), 2, 2 \rangle$.

(3) (10 points)

(a) Give a formula for the gradient vector, and describe its geometric properties.

(b) Find the gradient vector at the point $(2, -1, 1)$ for the function $f(x, y, z) = \cos(xy + 2z)$.

a) $f(x,y,z)$ has gradient $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

∇f points in the direction of fastest rate of change, and its length is equal to the fastest rate of change.

b) $\nabla f = \langle -\sin(xy+2z).y, -\sin(xy+2z).x, -\sin(xy+2z).2 \rangle$

$\nabla f(2,-1,1) = \langle 0, 0, 0 \rangle$

(4) (10 points) Find the critical points for the function $f(x, y) = x^2 + y^2 - \cancel{2}4xy + y$ and use the second derivative test to attempt to classify them.

$$\frac{\partial f}{\partial x} = 2x - 4y = 0 \qquad x = 2y$$

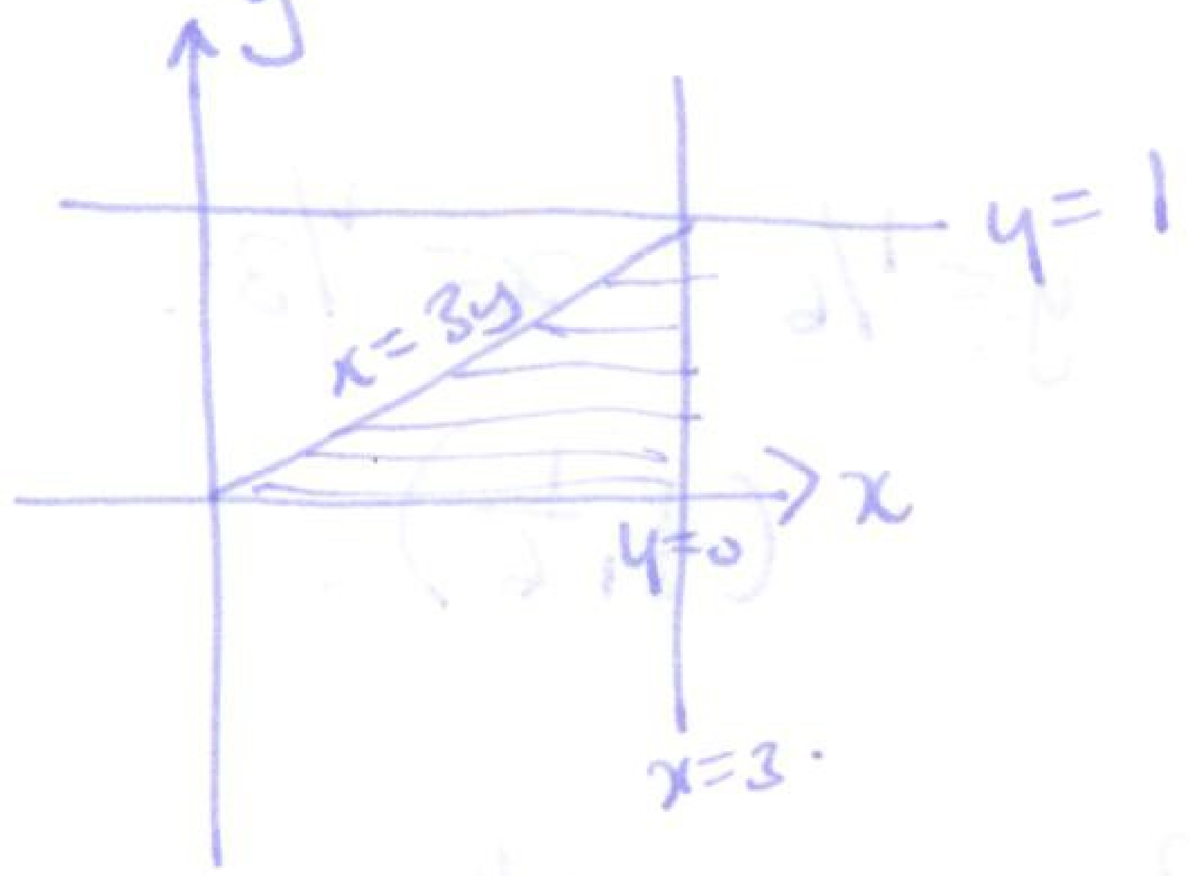
$$\frac{\partial f}{\partial y} = 2y - 4x + 1 = 0 \qquad -6y + 1 = 0 \qquad y = 1/6, \quad x = 1/3.$$

$$\left(\tfrac{1}{3}, \tfrac{1}{6}\right).$$

$$\left.\begin{aligned} f_{xx} &= 2 \\ f_{xy} &= -4 \\ f_{yy} &= 2 \end{aligned}\right\} \quad D = 4 - (-4)^2 = -12 \qquad \underline{\text{saddle}}.$$

(5) (10 points) Evaluate the following integral by changing the order of integration.

$$\int_0^1 \int_{3y}^3 e^{-x^2}\, dx dy$$



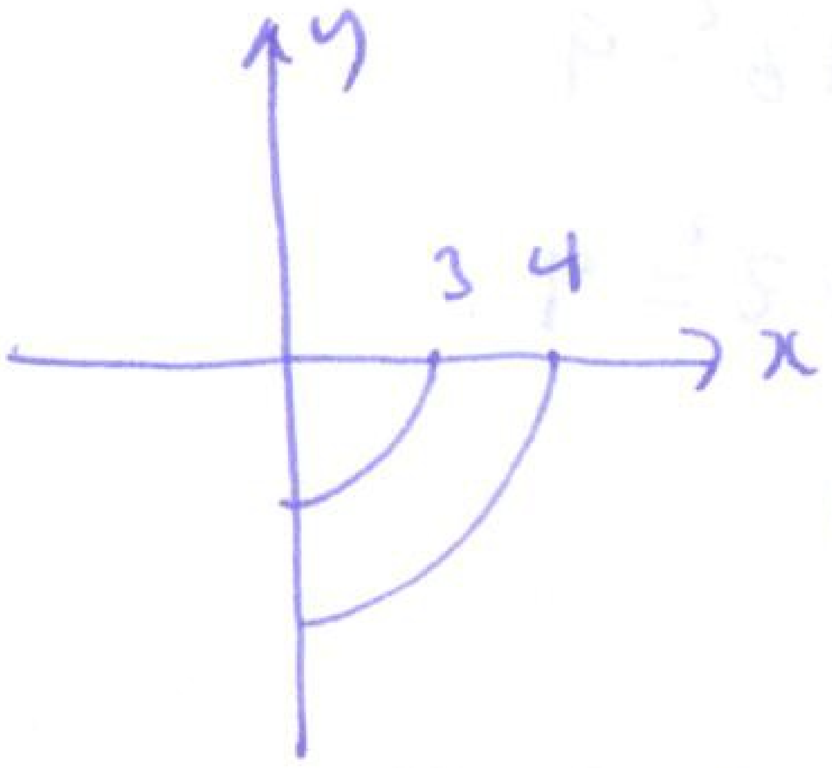
$$\int_0^3 \underbrace{\int_0^{x/3} e^{-x^2}\, dy}\, dx$$

$$\left[y e^{-x^2} \right]_0^{x/3} = \frac{x}{3} e^{-x^2}$$

$$\int_0^3 \frac{x}{3} e^{-x^2} dx = \left[-\frac{1}{6} e^{-x^2} \right]_0^3$$

$$= \frac{1}{6}\left(1 - e^{-9}\right)$$

(6) (10 points) Integrate the function $f(x, y) = -x^2 - y^2$ in the region between the circle of radius 3 and the circle of radius 4, which lies in the quadrant given by $x \geq 0, y \leq 0$. (Hint: use polar coordinates.)



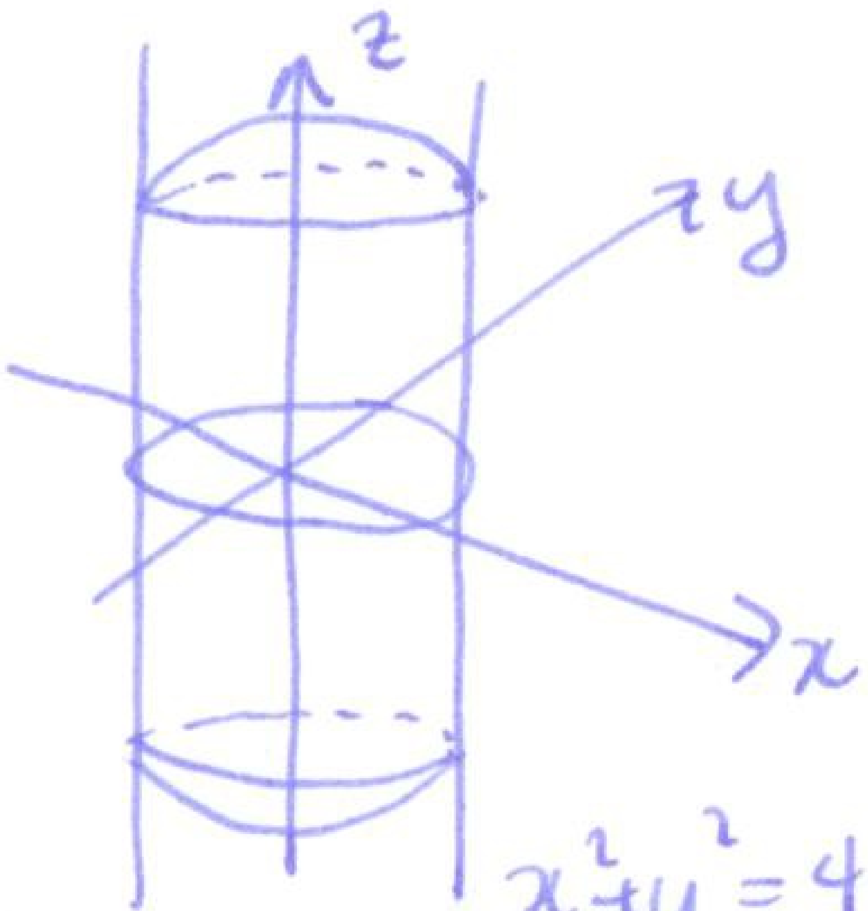
$$\int_{\frac{3\pi}{2}}^{2\pi} \int_{3}^{4} -r^2 \, r \, dr \, d\theta$$

$$\left[-\frac{1}{4} r^4\right]_3^4 = \frac{1}{4}(81-256)$$

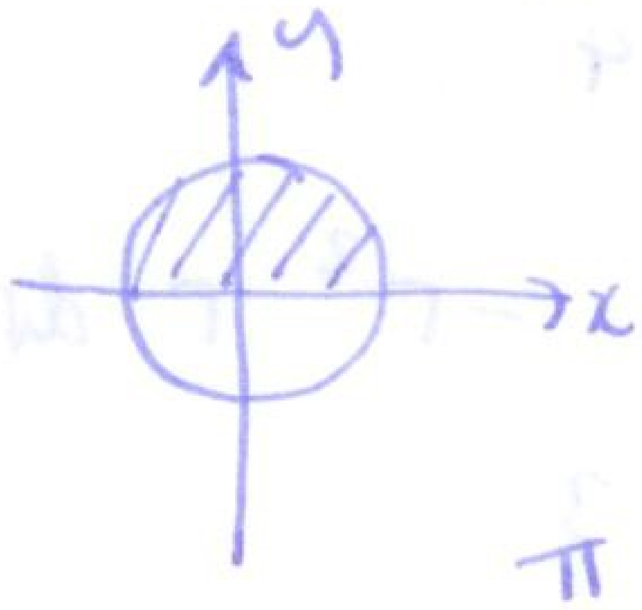
$$\frac{1}{4}(81-256)\left[\theta\right]_{\frac{3\pi}{2}}^{2\pi} = \frac{\pi}{8}(81-256)$$

(7) (10 points) Write down limits for an integral over the region inside the cylinder $x^2 + y^2 = 4$, with $y \geq 0$, and inside the sphere of radius 3. You may use any coordinate system.

use cylindrical coords



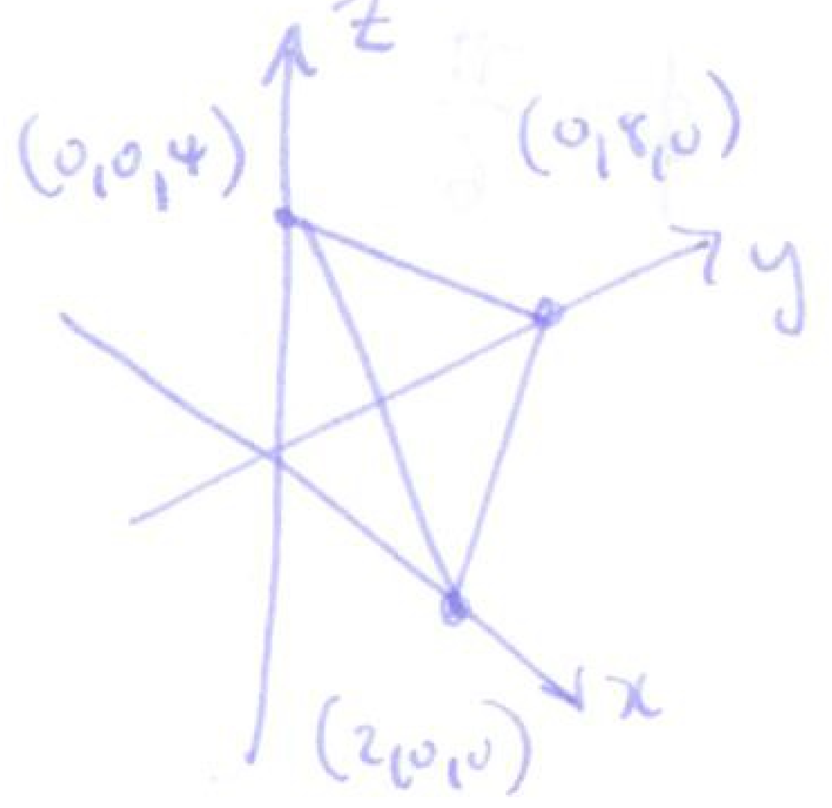
$x^2+y^2=4$
$\Updownarrow$
$r=2$

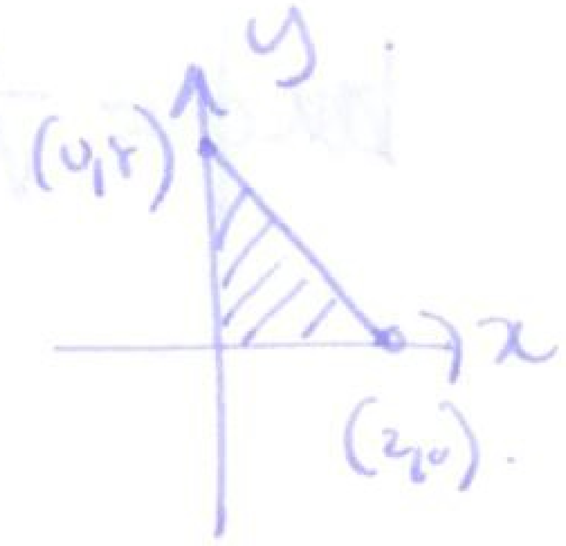


$x^2+y^2+z^2=9$
$\Updownarrow$
$r^2+z^2=9$

$$\int_0^{\pi} \int_0^{3} \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} f(-)\, r\, dz\, dr\, d\theta$$

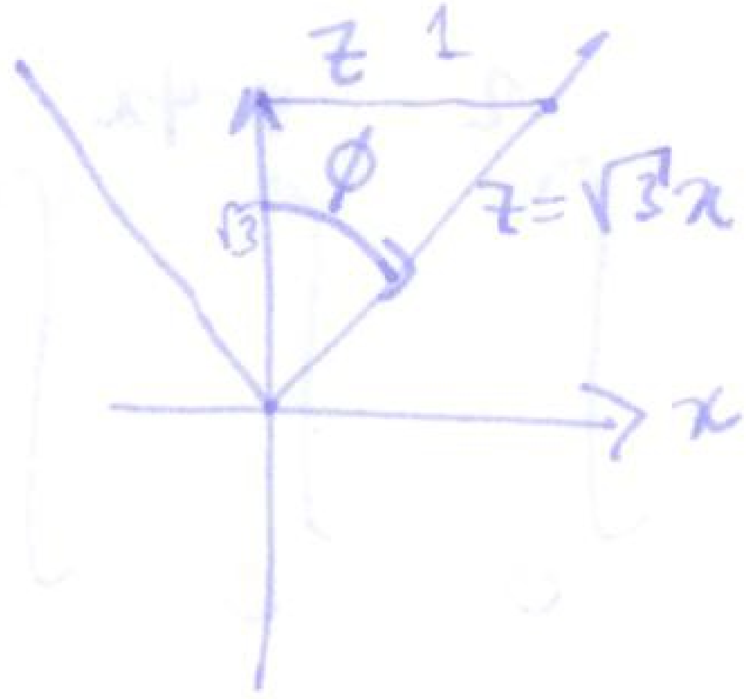
(8) (10 points) Write down limits for a 3-dimensional integral over the region in the positive octant (i.e. $x \geq 0, y \geq 0, z \geq 0$), below the plane $4x + y + 2z = 8$.





$$\int_0^2 \int_0^{8-4x} \int_0^{4-2x-y/2} f(-)\, dz\, dy\, dx$$

(9) (10 points) Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 9$, with $y \geq 0$, which lies above the cone $z = \sqrt{3(x^2 + y^2)}$.



$\tan\phi = \frac{1}{\sqrt{3}}$, $\phi = \frac{\pi}{6}$.

$$\int_0^{\pi/6} \int_0^{3} \int_0^{\pi} \rho^2 \sin\phi \, d\theta \, d\rho \, d\phi$$

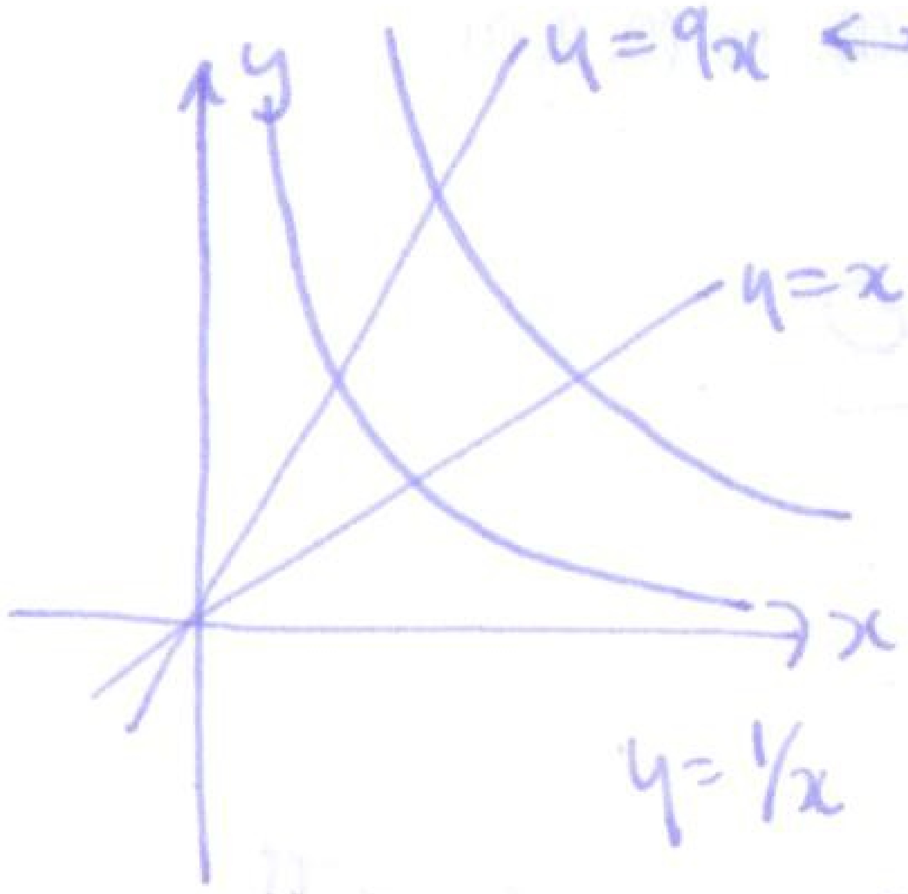
$\int_0^{2\pi} d\theta = \pi$.

$$\int_0^3 \rho^2 \, d\rho = \left[\frac{1}{3}\rho^3\right]_0^3 = 9$$

$$\int_0^{\pi/6} \sin\phi \, d\phi = \left[-\cos\phi\right]_0^{\pi/6} = -\cos\left(\frac{\pi}{6}\right) + 1 = -\frac{\sqrt{3}}{2} + 1.$$

$$V = 9\pi\left(1 - \frac{\sqrt{3}}{2}\right)$$

(10) Use the change of variable given by $x = v/u, y = uv$ to evaluate the integral $\displaystyle\iint_R \frac{1}{x}\,dxdy$, where R is the region bounded by the lines $y = 1/x$, $y = 9/x$, $y = x$ and $y = 9x$.

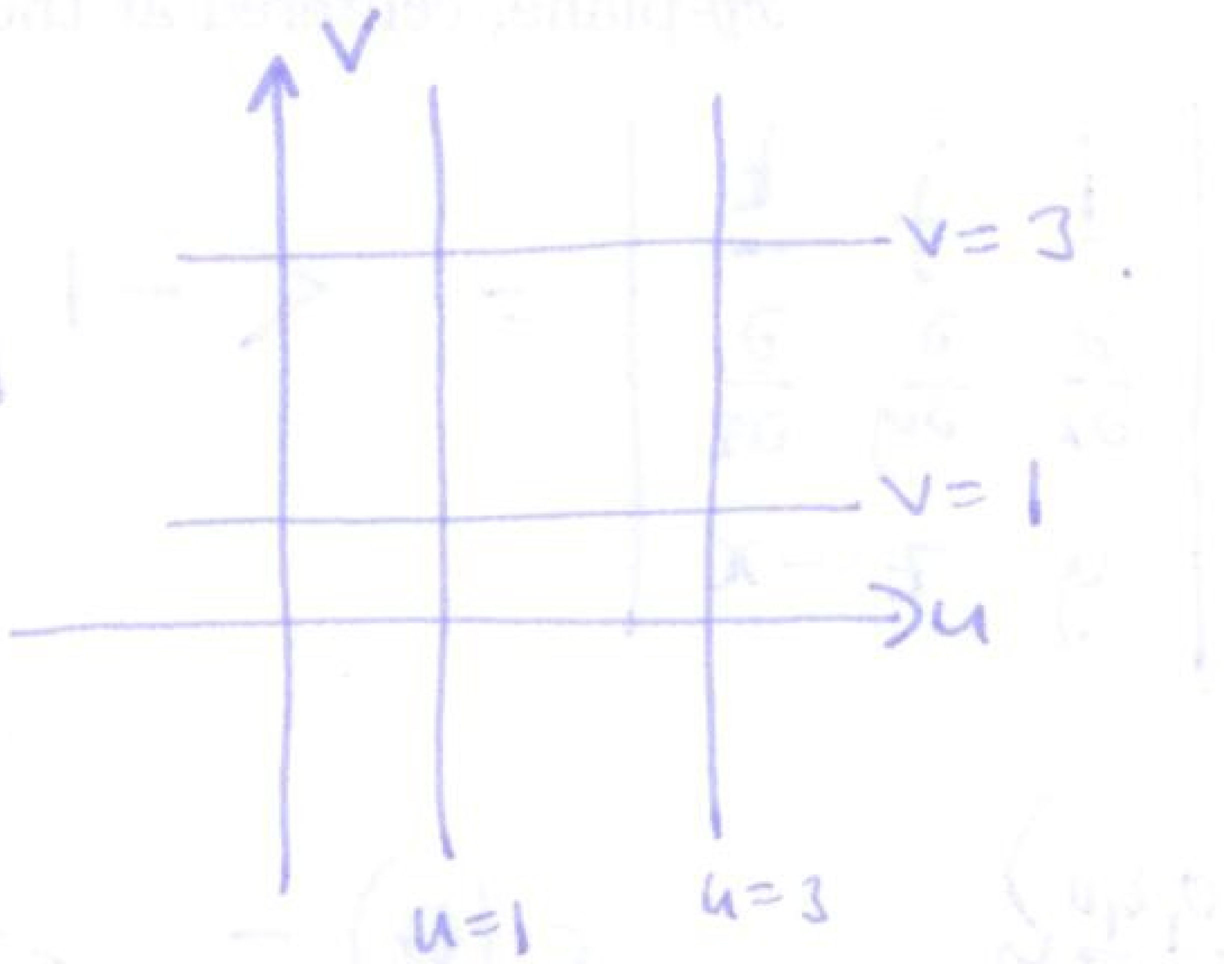


$y = 9x \leftrightarrow uv = \frac{9v}{u} \leftrightarrow u^2 = 9$
$u = 3.$

$y = x \leftrightarrow uv = \frac{v}{u} \leftrightarrow u = 1$

$y = 9/x \leftrightarrow uv = \frac{9u}{v}$
$\updownarrow$
$v^2 = 9$
$\updownarrow$
$v = 3$

$y = 1/x$
$\leftrightarrow v = 1$



$$\int_1^3\int_1^3 \frac{u}{v}\,|J|\,du\,dv \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ v & u \end{vmatrix}$$

$$= -\frac{v}{u} - \frac{v}{u} = -\frac{2v}{u}$$

$$\int_1^3\int_1^3 \frac{u}{v}\,\frac{2v}{u}\,du\,dv = \int_1^3\int_1^3 2\,du\,dv = 8$$

(11) (10 points)

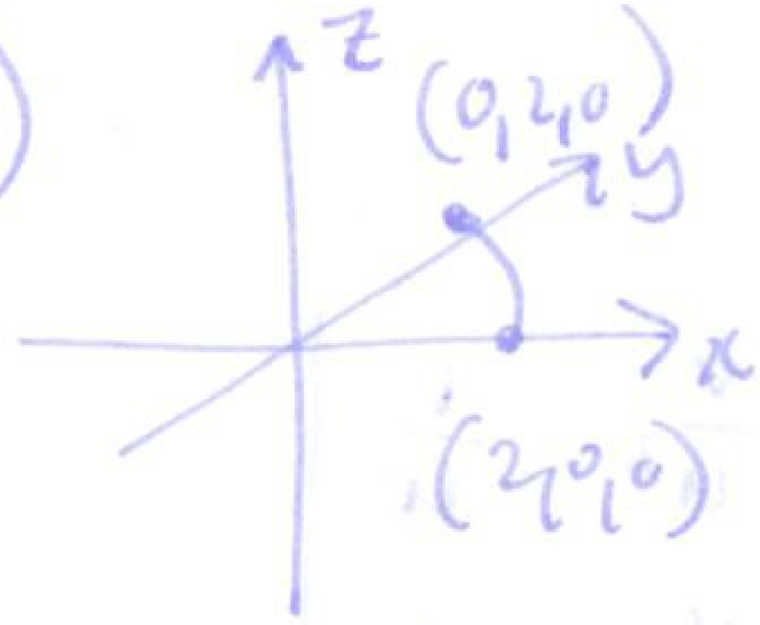
(a) Is the vector field $\mathbf{F} = \langle y, z, -x \rangle$ conservative? If so, find the potential function.

(b) Evaluate $\int_C \mathbf{F}.d\mathbf{s}$, where C is the arc of the circle of radius 2 in the xy-plane, centered at the origin, which goes from $(2, 0, 0)$. to $(0, 2, 0)$.

a) $$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & -x \end{vmatrix} = \langle -1, 1, -1 \rangle \neq \underline{0}.$$

not conservative.

b)



$$\underline{c}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0 \rangle \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\underline{c}'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$$

$$\int_C \underline{F}(\underline{c}(\theta)).\underline{c}'(\theta)\, d\theta = \int_0^{\pi/2} \langle 2\sin\theta, 0, -2\cos\theta \rangle . \langle -2\sin\theta, 2\cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{\pi/2} -4\sin^2\theta \, d\theta = \int_0^{\pi/2} 2 - 2\cos 2\theta \, d\theta = \Big[2\theta - \sin 2\theta \Big]_0^{\pi/2}$$

$$= \pi$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$

(12) (10 points)

(a) Is the vector field $\mathbf{F} = \langle -y, -x + z, y \rangle$ conservative? If so, find the potential function.

(b) Evaluate $\int_C \mathbf{F}.d\mathbf{s}$, where C is the helix of radius 1 which rotates twice around the ~~origin~~ x-axis between $(1, 0, 0)$ and $(8, 0, 0)$.

a) $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & -x+z & y \end{vmatrix} = \langle 1-1, 0, -1+1 \rangle = \underline{0}$

$$\left.\begin{aligned} \int -y\,dx &= -xy + c_1(y,z) \\ \int -x+z\,dy &= -xy + zy + c_2(x,z) \\ \int y\,dz &= yz + c_3(x,y) \end{aligned}\right\} \quad f(x,y,z) = -xy + yz$$

b) $\int_C F.ds = f(8,0,0) - f(1,0,0) = 0.$