Math 233 Calculus 3 Spring 12 Final a

Name: Solutions

- Do any 10 of the following 12 questions.
- You may use a calculator, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Midterm 3	
Overall	

(1) (10 points) Let **u** be the vector $\langle 2, -1, 2\rangle$, and let **v** be the vector $\langle 2, 1, -1\rangle$.

(a) Write **v** as the sum of two vectors, one parallel to **u**, and one perpendicular to **u**.

(b) Find the area of the triangle formed by the two vectors **u** and **v**.

a) $\frac{\underline{u}.\underline{v}}{\underline{u}.\underline{u}}\underline{u} = \frac{4-1-2}{4+1+4}\underline{u} = \frac{1}{9}\langle 2,-1,2\rangle$

$\underline{v} - \frac{1}{9}\langle 2,-1,2\rangle = \langle \frac{16}{9}, \frac{10}{9}, \frac{-11}{9}\rangle$

$\underline{v} = \langle \frac{2}{9}, \frac{-1}{9}, \frac{2}{9}\rangle + \langle \frac{16}{9}, \frac{10}{9}, \frac{-11}{9}\rangle$

parallel to $\underline{u}$ perpendicular to $\underline{u}$

b) $\underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \langle -1, 6, 4\rangle$

area $= \frac{1}{2}\|\underline{u}\times\underline{v}\| = \frac{1}{2}\sqrt{1+36+16} = \frac{1}{2}\sqrt{53}$

(2) (10 points)

(a) Give an example of a parameterized curve in $\mathbb{R}^3$ which has constant speed, but not constant velocity.

(b) A particle starts at the origin at time 0, and has velocity given by $\mathbf{r}'(t) = \langle e^{-2t}, t, 1\rangle$. Where is it at time $t = 4$?

a) $\underline{r}(t) = \langle \cos t, \sin t, 0 \rangle \quad \underline{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$ not constant

$\|\underline{r}'(t)\| = 1$

b) $\underline{r}(t) = \langle -\frac{1}{2}e^{-2t}, \frac{1}{2}t^2, t \rangle + \underline{c}$

$\underline{r}(0) = \langle -\frac{1}{2}, 0, 0 \rangle + \underline{c} = \underline{0} \Rightarrow \underline{c} = \langle \frac{1}{2}, 0, 0 \rangle$

$\underline{r}(4) = \langle \frac{1}{2} - \frac{1}{2}e^{-8}, 8, 4 \rangle.$

(3) (10 points)

(a) Give a formula for the gradient vector, and describe its geometric properties.

(b) Find the gradient vector at the point $(2, -1, 1)$ for the function $f(x, y, z) = \sin(xy - 2z)$.

a) $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

∇f points in the direction of greatest rate of increase, length is greatest rate of increase.

b) $\nabla f = \langle \cos(xy-2z) \cdot y, \cos(xy-2z) \cdot x, \cos(xy-2z) \cdot (-2) \rangle$

$\nabla f(2,-1,1) = \langle -\cos(4), 2\cos(4), -2\cos(4) \rangle$.

(4) (10 points) Find the critical points for the function $f(x, y) = x^2 + y^2 - 3xy + x$ and use the second derivative test to attempt to classify them.

$\frac{\partial f}{\partial x} = 2x - 3y + 1 = 0$

$\frac{\partial f}{\partial y} = 2y - 3x = 0$

$y = \frac{3}{2}x$

$2x - \frac{9}{2}x + 1 = 0$

$x = \frac{2}{5}$

$y = \frac{3}{5}$

critical point $(\frac{2}{5}, \frac{3}{5})$.

$f_{xx} = 2$

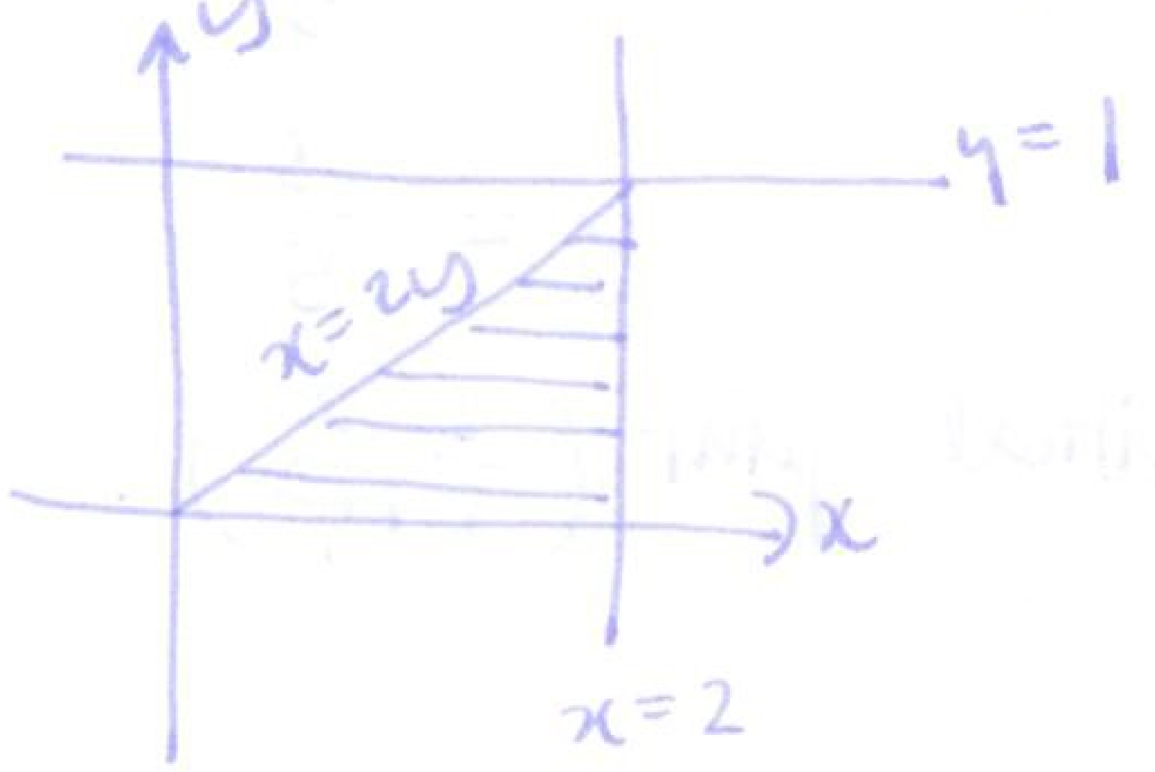
$f_{xy} = -3$

$f_{yy} = 2$

$D = 4 - 9 = -5$ saddle

(5) (10 points) Evaluate the following integral by changing the order of integration.

$$\int_0^1 \int_{2y}^2 e^{-x^2}\, dx dy$$

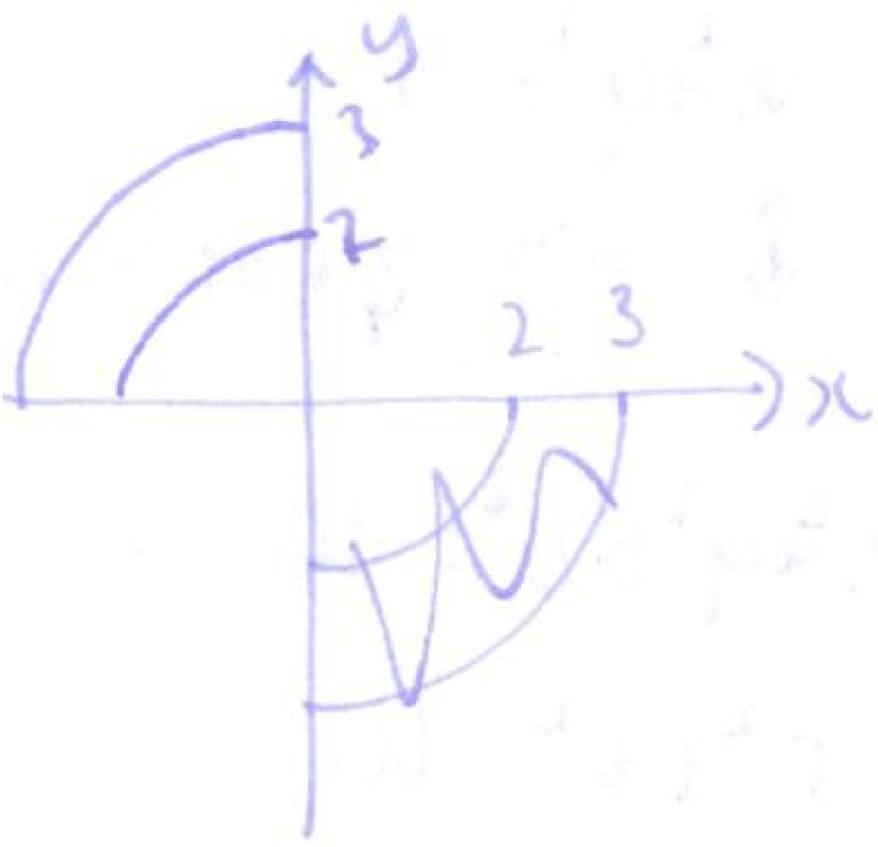


$$\int_0^2 \underbrace{\int_0^{x/2} e^{-x^2}\, dy}\, dx$$

$$\left[y e^{-x^2} \right]_0^{x/2} = \frac{x}{2} e^{-x^2}$$

$$\int_0^2 \frac{x}{2} e^{-x^2} dx = \left[-\frac{1}{4} e^{-x^2} \right]_0^2 = \frac{1}{4}\left(1 - e^{-4}\right)$$

(6) (10 points) Integrate the function $f(x,y) = x^2 + y^2$ in the region between the circle of radius 2 and the circle of radius 3, which lies in the quadrant given by $x \leq 0, y \geq 0$. (Hint: use polar coordinates.)



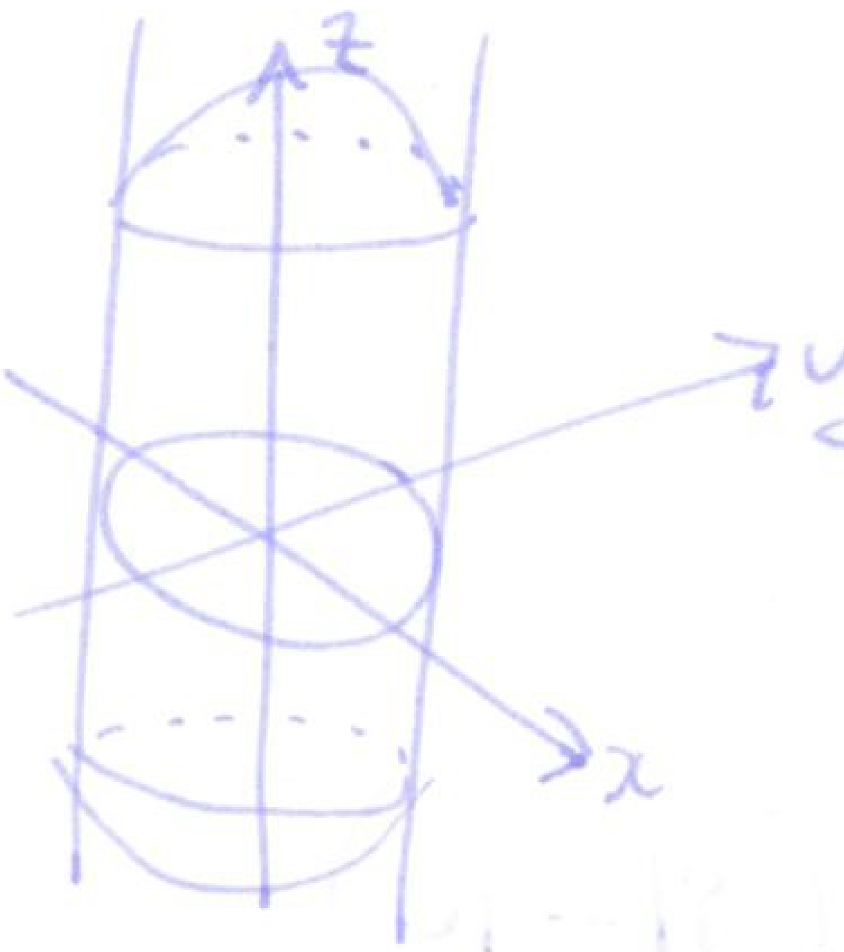
$$\int_{\frac{\pi}{2}}^{\pi} \int_2^3 r^2 \; r \, dr \, d\theta$$

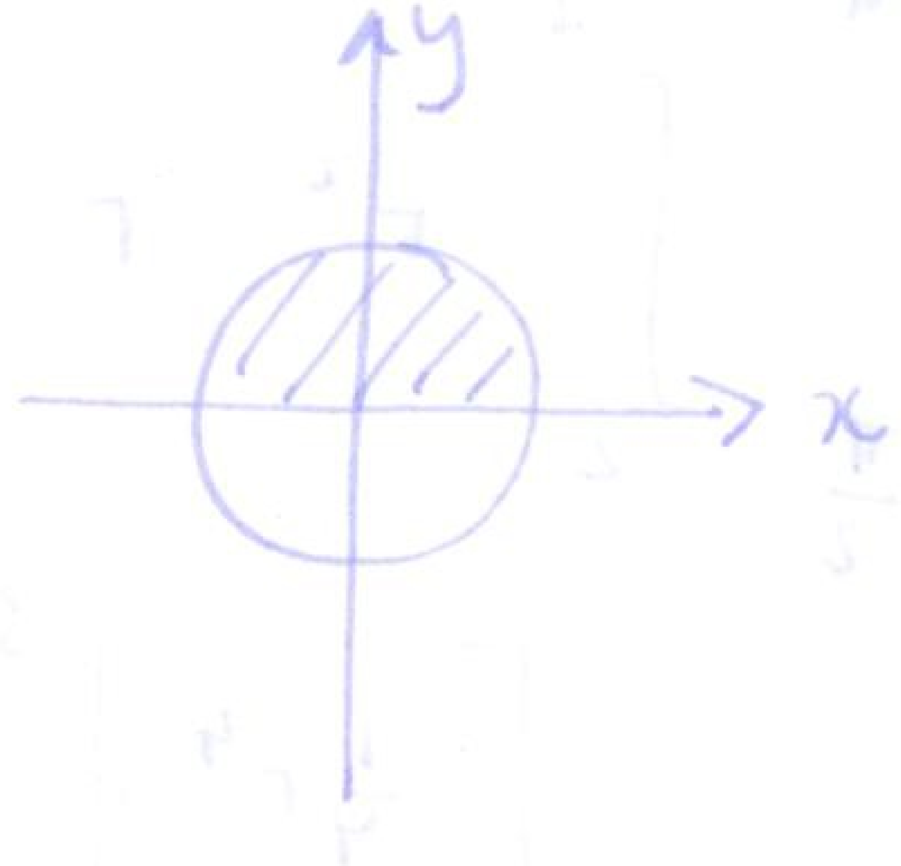
$$\left[\frac{1}{4} r^4\right]_2^3 = \frac{1}{4}(81-16)$$

$$= \frac{\pi}{8}(81-16)$$

(7) (10 points) Write down limits for an integral over the region inside the cylinder $x^2 + y^2 = 4$, with $y \geq 0$, and inside the sphere of radius 4. You may use any coordinate system.

use cylindrical coordinates





$x^2+y^2=4$

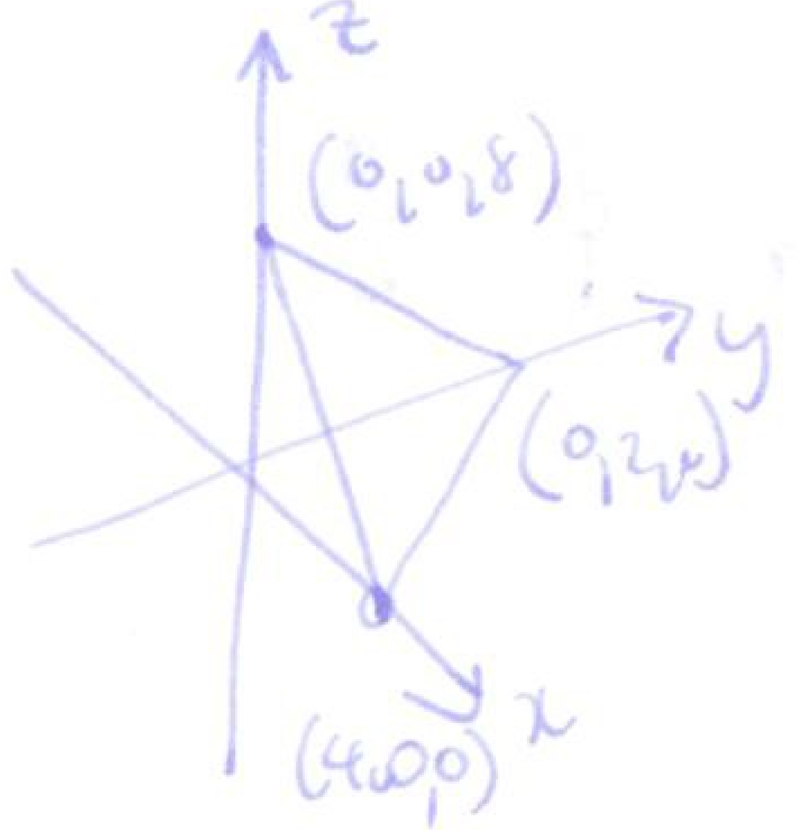
$\updownarrow \quad r^2 = 4 \Leftrightarrow r = 2.$

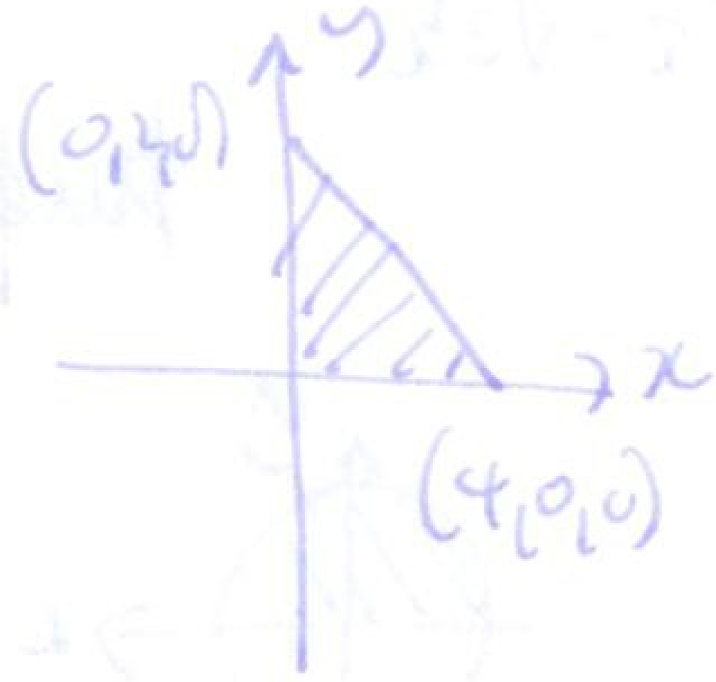
$x^2+y^2+z^2=16$

$\updownarrow \quad r^2+z^2=16$

$$\int_0^{\pi}\int_0^{2}\int_{-\sqrt{16-r^2}}^{+\sqrt{16-r^2}} f(\ ,\ )\, r\, dz\, dr\, d\theta$$

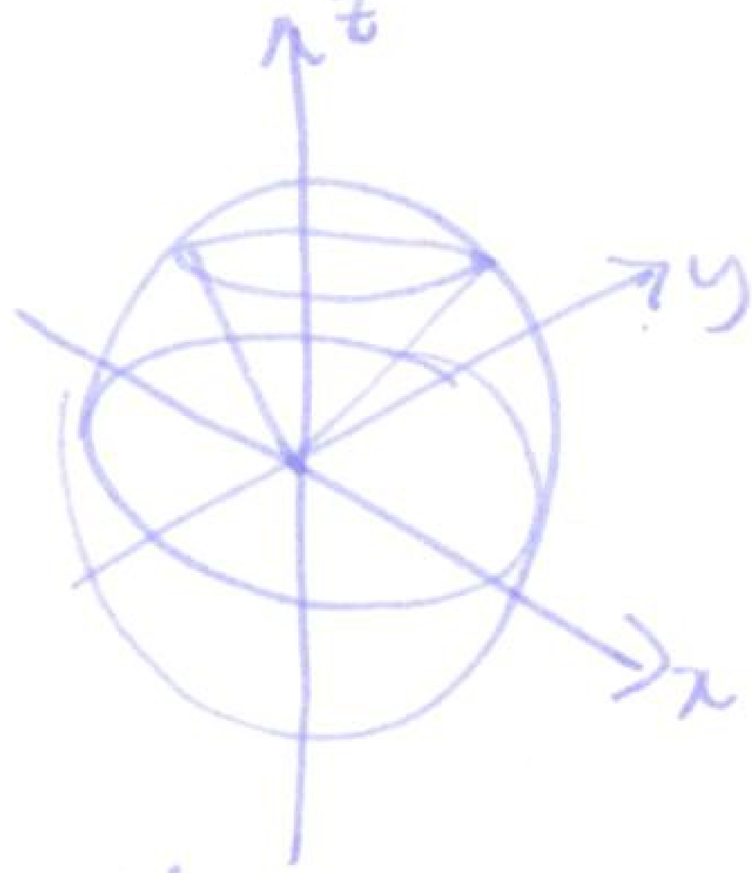
(8) (10 points) Write down limits for a 3-dimensional integral over the region in the positive octant (i.e. $x \geq 0, y \geq 0, z \geq 0$), below the plane $2x + 4y + z = 8$.

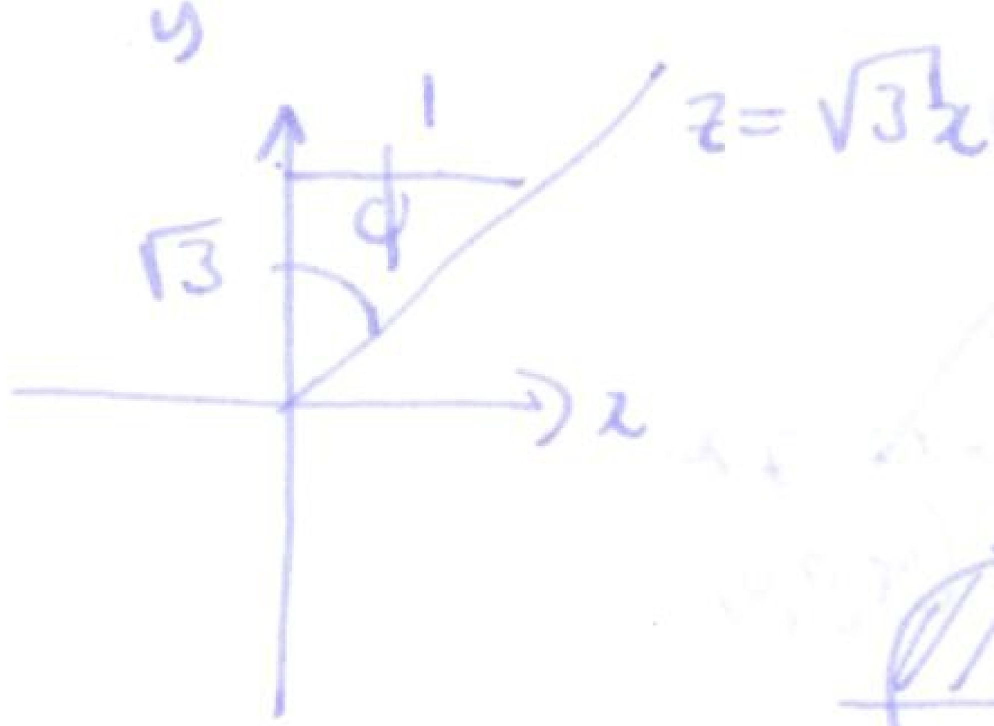




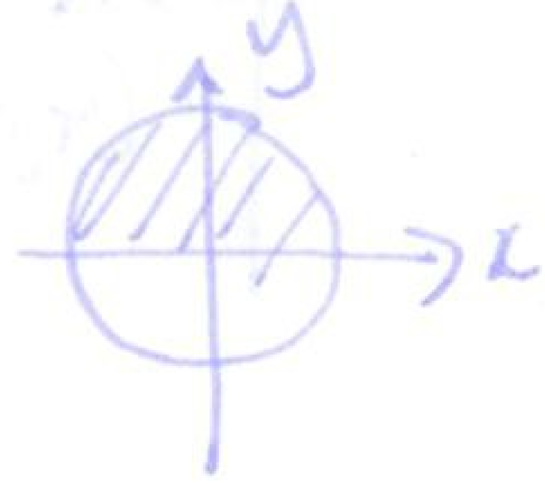
$$\int_0^4 \int_0^{2-\frac{x}{2}} \int_0^{8-2x-4y} f(-)\, dz\, dy\, dx$$

(9) (10 points) Find the volume of the region inside the sphere $x^2+y^2+z^2=4$, with $y \geq 0$, which lies above the cone $z=\sqrt{3(x^2+y^2)}$.





$\tan\phi = \frac{1}{\sqrt{3}}$ $\phi = \frac{\pi}{6}$.



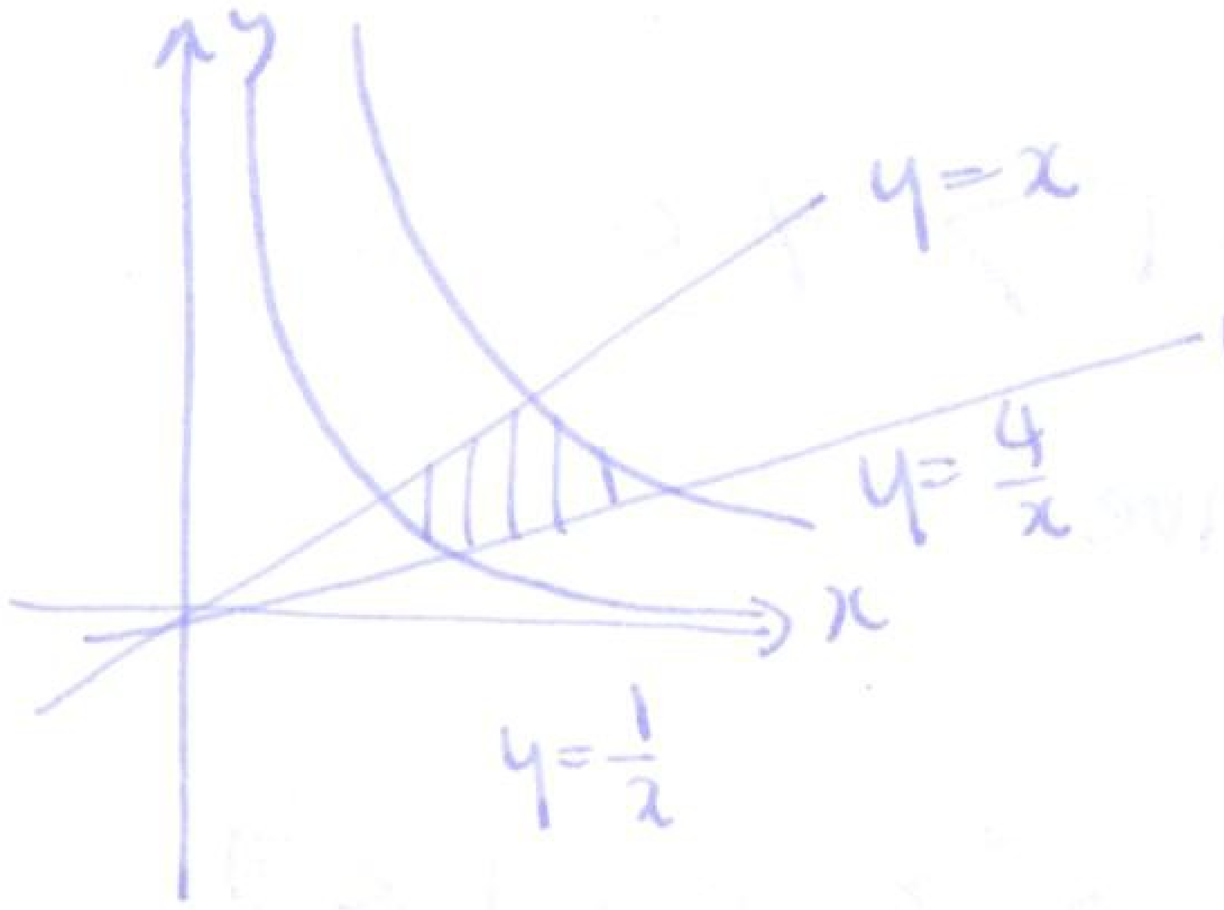
$$\int_0^{\pi/6}\int_0^{\pi}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

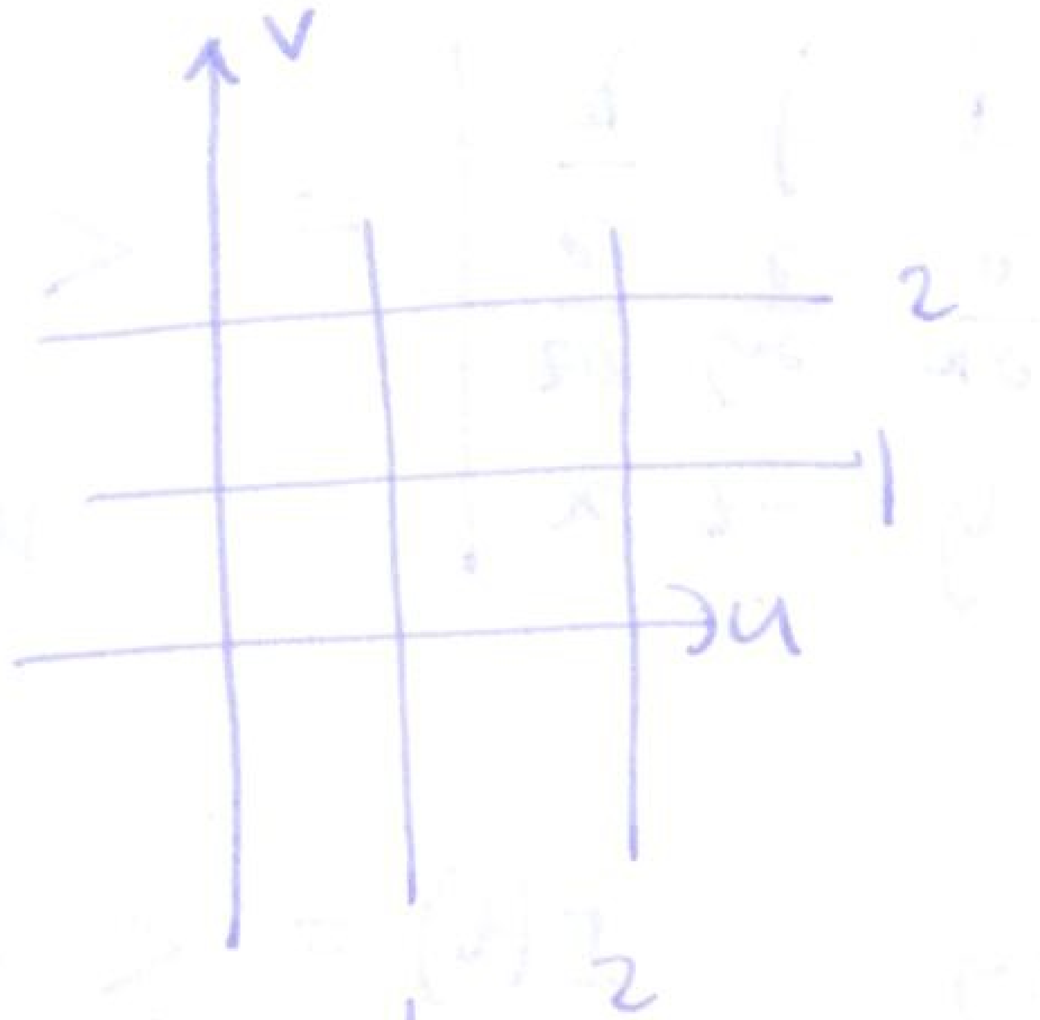
$$\int_0^2 \rho^2 \, d\rho = \left[\tfrac{1}{3}\rho^3\right]_0^2 = \frac{8}{3} \qquad \int_0^{\pi} d\theta = \pi$$

$$\int_0^{\pi/6} \sin\phi \, d\phi = \left[-\cos\phi\right]_0^{\pi/6} = -\frac{\sqrt{3}}{2}+1$$

volume $= \frac{8\pi}{8}\left(1-\frac{\sqrt{3}}{2}\right)$

(10) Use the change of variable given by $x = uv, y = u/v$ to evaluate the integral $\int\int_R \frac{1}{y}\, dxdy$, where R is the region bounded by the lines $y = 1/x$, $y = 4/x$, $x = y$ and $x = 4y$.





$y = x \Leftrightarrow \frac{u}{v} = uv \Leftrightarrow v^2 = 1 \Leftrightarrow v = 1$

$y = \frac{x}{4} \Leftrightarrow \frac{u}{v} = \frac{uv}{4} \Leftrightarrow v^2 = 4 \Leftrightarrow v = 2$

$xy = 1 \Leftrightarrow u^2 = 1 \Leftrightarrow u = 1$

$xy = 4 \Leftrightarrow u^2 = 4 \Leftrightarrow u = 2$

$$\int_1^2 \int_1^2 \frac{v}{u} |J|\, dudv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}$$

$$\int_1^2 \int_1^2 \frac{v}{u} \frac{2u}{v}\, dudv = \int_1^2 \int_1^2 2\, dudv = 2.$$

(11) (10 points)

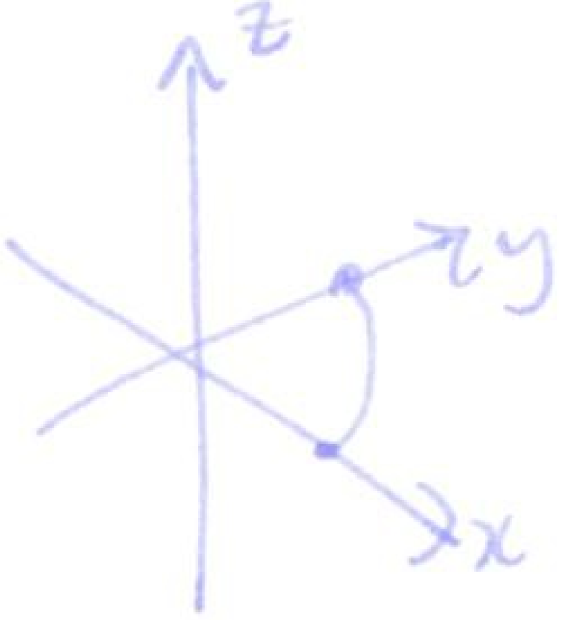
(a) Is the vector field $\mathbf{F} = \langle y, -z, x\rangle$ conservative? If so, find the potential function.

(b) Evaluate $\int_C \mathbf{F}.d\mathbf{s}$, where C is the arc of the circle of radius 3 in the xy-plane, centered at the origin, which goes from $(3,0,0)$. to $(0,3,0)$.

a) $\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -z & x \end{vmatrix} = \langle 1, -1, -1\rangle \neq \underline{0}$

not conservative

b)



$\underline{r}(t) = \langle 3\cos t, 3\sin t, 0\rangle \quad 0 \le t \le \frac{\pi}{2}$

$\underline{r}'(t) = \langle -3\sin t, 3\cos t, 0\rangle$

$\int_C \underline{F}.\underline{ds} = \int_0^{\pi/2} \langle 3\sin t, 0, 3\cos t\rangle . \langle -3\sin t, 3\cos t, 0\rangle \, dt$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$= \int_0^{\pi/2} -9\sin^2 t \, dt$

$= \int_0^{\pi/2} -9\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt = -\frac{9}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi/2}$

$= -\frac{9\pi}{4}$

(12) (10 points)

(a) Is the vector field $\mathbf{F} = \langle y, x-z, -y \rangle$ conservative? If so, find the potential function.

(b) Evaluate $\int_C \mathbf{F}.d\mathbf{s}$, where C is the helix of radius 1 which rotates thtree times around the origin between $(1,0,0)$ and $(9,0,0)$.

a) $\nabla \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-z & -y \end{vmatrix} = \langle -1+1, 0, 1-1 \rangle = \underline{0}$

conservative.

$$\left.\begin{aligned} \int y\,dx &= xy + c_1(y,z) \\ \int x-z\,dy &= xy - zy + c_2(x,z) \\ \int -y\,dz &= -yz + c_3(x,y) \end{aligned}\right\} f(x,y,z) = xy - zy$$

b) $\int_C \underline{F}.\underline{ds} = f(1,0,0) - f(9,0,0) = 0$.