Math 231 Calculus 1 Spring 2012

FINAL EXAM b

Name: Solutions

ANSWER ALL QUESTIONS IN THE SPACE PROVIDED

Please present clear solutions and fully explain your reasoning in complete sentences. Answers submitted without justification will not receive full credit.

Do all questions in Part I.

Do any two questions in Part II. You must label the problems to be graded, otherwise the first two problems attempted will be graded.

Only a calculator is permitted.

GOOD LUCK!

Part I			Part II		
1	12		1	10	
2	12		2	10	
3	12		3	10	
4	12		4	10	
5	12		5	10	
6	10		Total II	20	
7	10				
Total I	80		Total	100	

Part I (Do all questions in this part.)

(1) (12 points) Differentiate the following functions. Do not simplify your answers.

(a) $g(x) = \dfrac{\ln x}{x-1}$

$$\frac{(x-1)\left(\frac{1}{x}\right) - \ln x}{(x-1)^2}$$

(b) $f(x) = e^x \sin(2x)$

$$e^x \sin(2x) + e^x \cdot 2\cos(2x)$$

(c) $h(x) = \sqrt{e^{3x} + 1}$

$$\frac{1}{2}\left(e^{3x}+1\right)^{-1/2} 3e^{3x}$$

(2) (12 points) Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^2+x-1}{\sqrt{x}}\,dx$ $= \displaystyle\int x^{3/2} + x^{1/2} - x^{-1/2}\,dx$

$= \dfrac{2x^{5/2}}{5} + \dfrac{2x^{3/2}}{3} - 2x^{1/2} + c$

(b) $\displaystyle\int \cos^2 x \sin x\; dx$ $\qquad -\cos^3 x + c$

(c) $\displaystyle\int_{-1}^{1} e^{-2x}\,dx$ $\qquad -\frac{1}{2}e^{-2x} + c$

(3) (12 points) Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x\to-2} \frac{x^2-x-6}{x+2}$. $= \lim_{x\to-2} \frac{(x+2)(x-3)}{(x+2)} = \lim_{x\to-2} x-3 = -5$

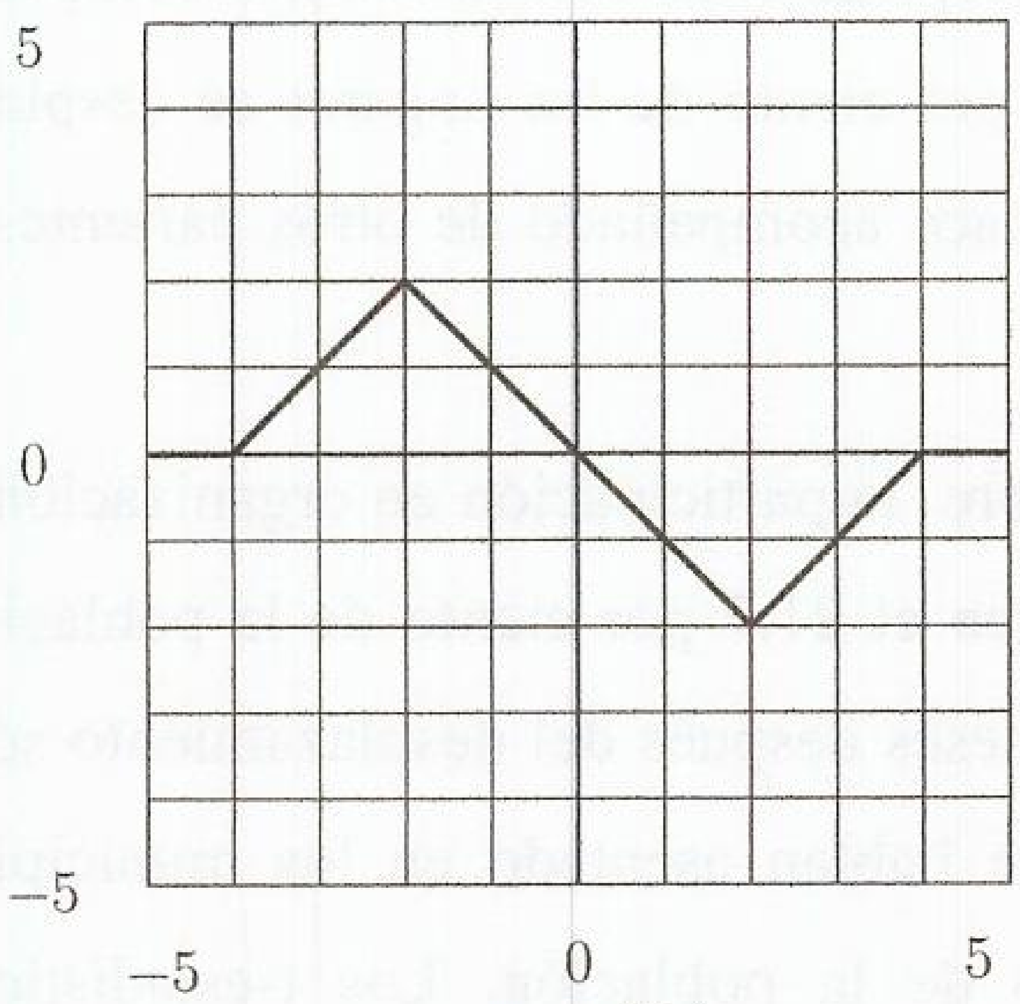
(b) Find $\lim_{x\to0} \frac{\sin 4x}{e^x-1}$. $\underset{\text{l'Hôpital}}{=} \frac{4\cos 4x}{e^x} = 4$

(c) Find $\lim_{x\to0} \frac{1}{x} - \frac{1}{\sin x}$. $= \lim_{x\to0} \frac{\sin x - x}{x\sin x}$

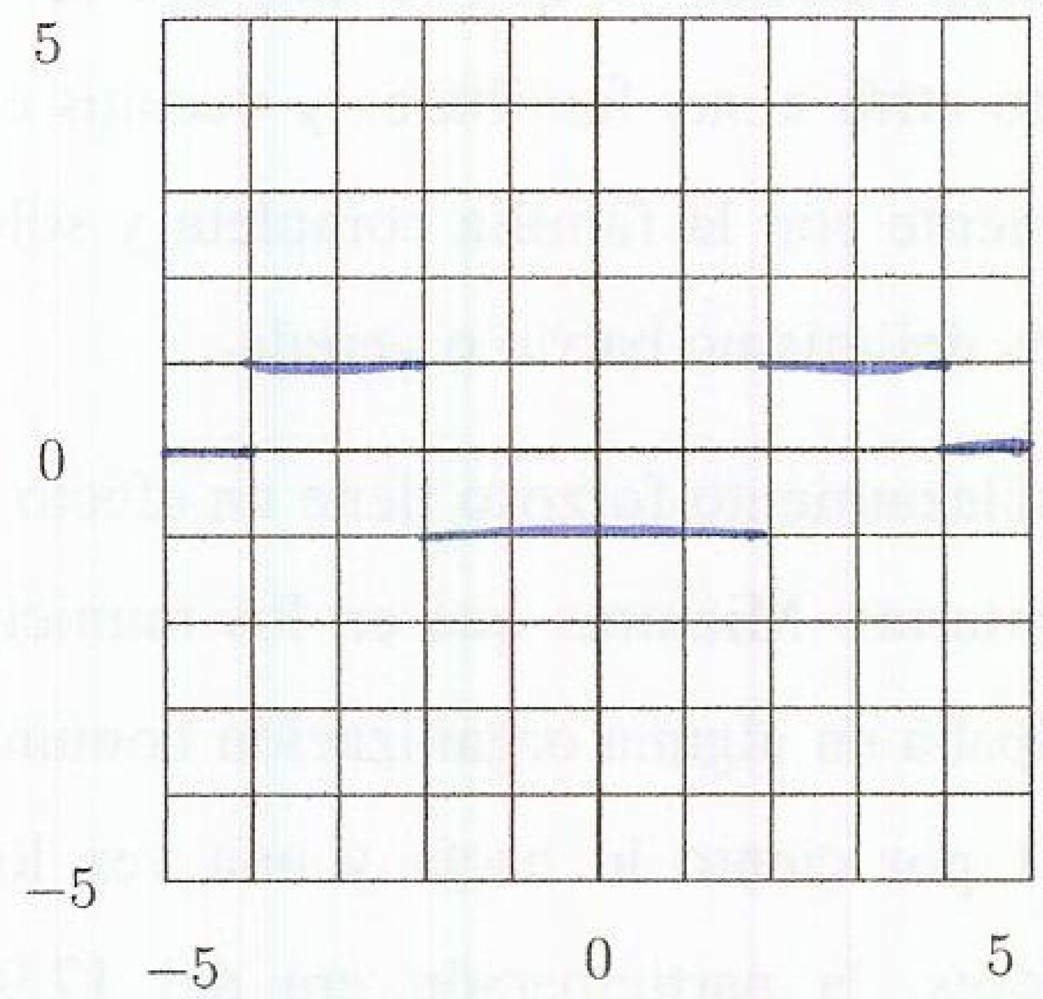
l'Hôpital: $\lim_{x\to0} \frac{\cos x - 1}{\sin x + x\cos x} \underset{\text{l'Hôpital}}{=} \lim_{x\to0} \frac{-\sin x}{\cos x + \cos x - x\sin x} = 0$

(4) (12 points)

(a) Sketch $y = f'(x)$ on the right hand graph.

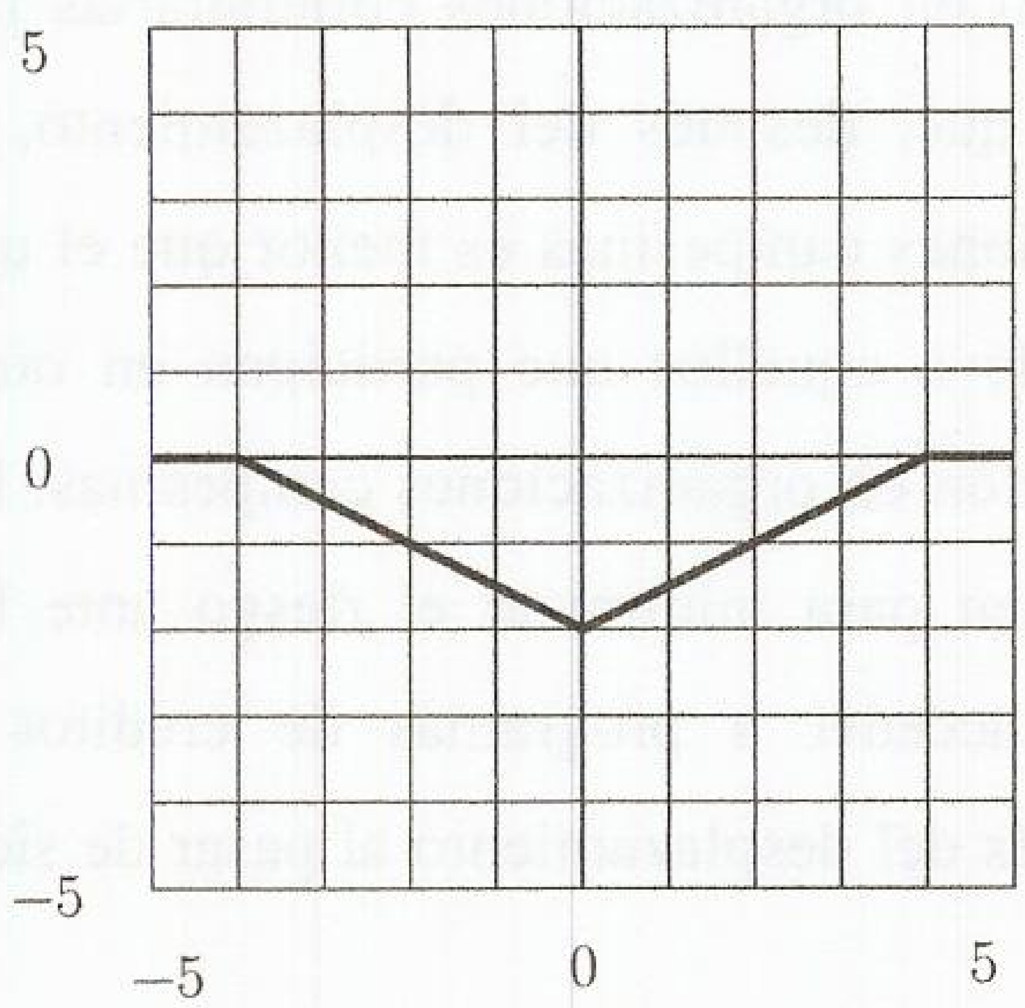


$y = f(x)$



$y = f'(x)$

(b) Find $\int_{-5}^{5} g(x)\, dx$, for the function drawn below.



-8

$y = g(x)$

(5) (12 points) Consider $f(x) = x^3 - 12x$.

(a) Find the critical points for $f(x)$.

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$ critical points $x = \pm 2$

(b) Give the intervals for which f is increasing, and intervals for which it is decreasing.

$f'(x)$: $+$ | -2 | $-$ | $+2$ | $+$

increasing $(-\infty, -2) \cup (2, \infty)$

decreasing $(-2, 2)$

(c) Give the intervals for which f is concave up, and for which it is concave down.

$f''(x) = 6x$

$f''(x)$: $-$ | 0 | $+$

concave up $(0, \infty)$

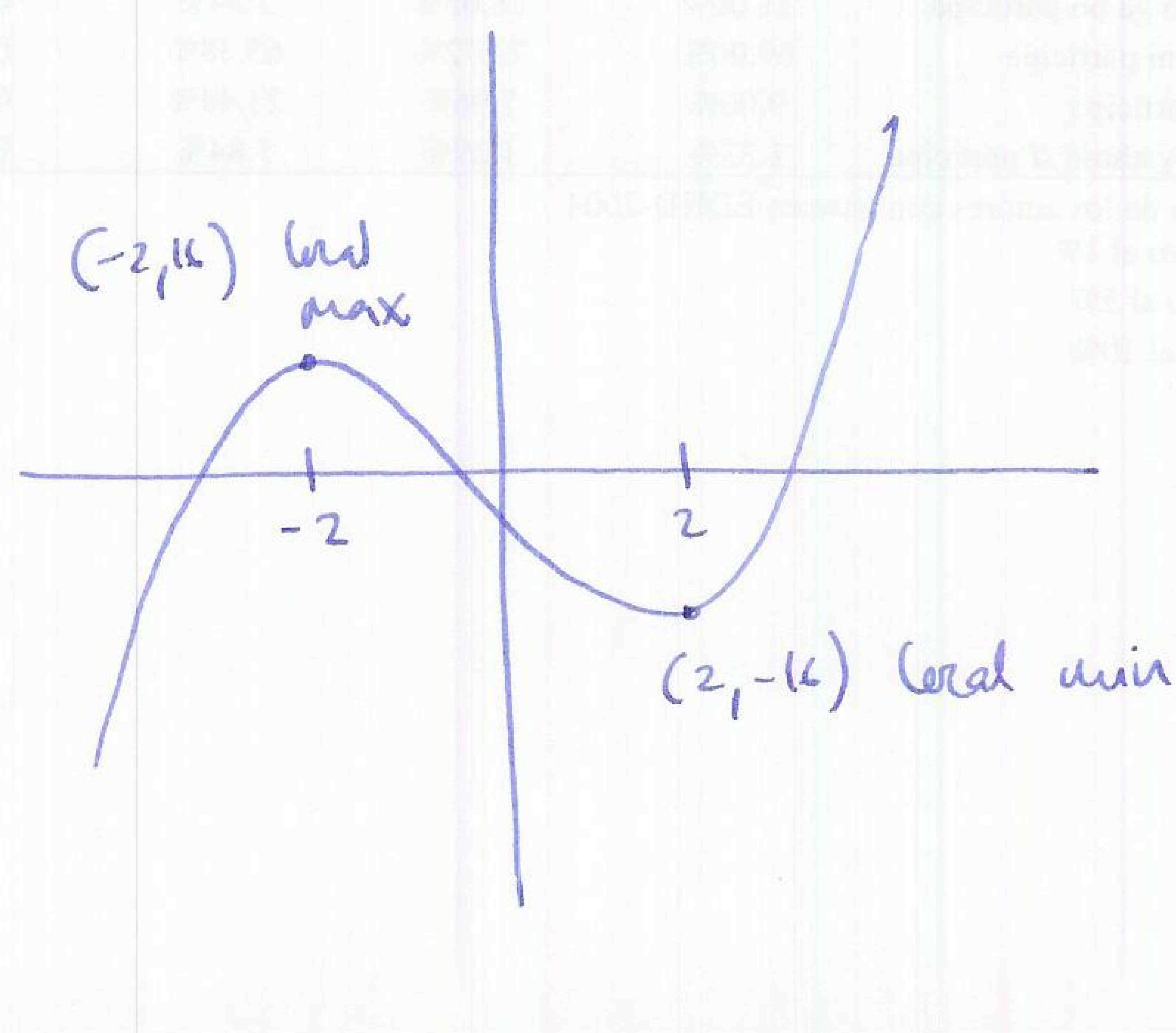
concave down $(-\infty, 0)$.

(d) Decide which critical points are maxima, minima, or neither.

$x = +2$ local min

$x = -2$ local max

(e) Sketch the graph of $f(x)$.

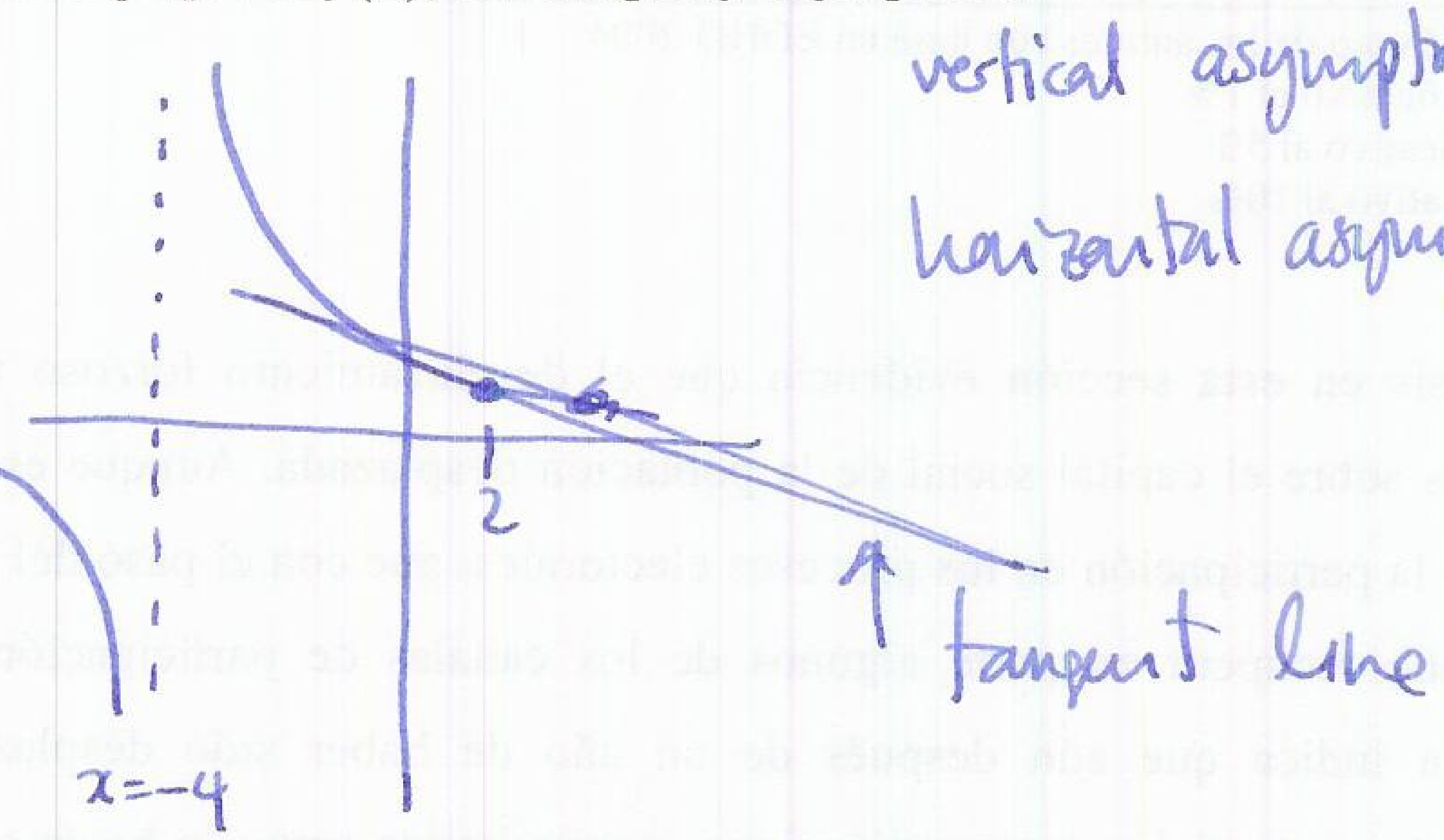


(6) (10 points) Consider $f(x) = \dfrac{2}{x+4}$.

(a) Sketch the graph of $f(x)$, showing any asymptotes.

vertical asymptote at $x = -4$

horizontal asymptotes $y=0$



(b) Find the slope of the tangent line at $x = 2$ using the derivative rules, and write down the equation for the tangent line.

$$f'(x) = -2(x+4)^{-2}$$

$$f'(2) = \frac{2}{36} = -\frac{1}{18}$$

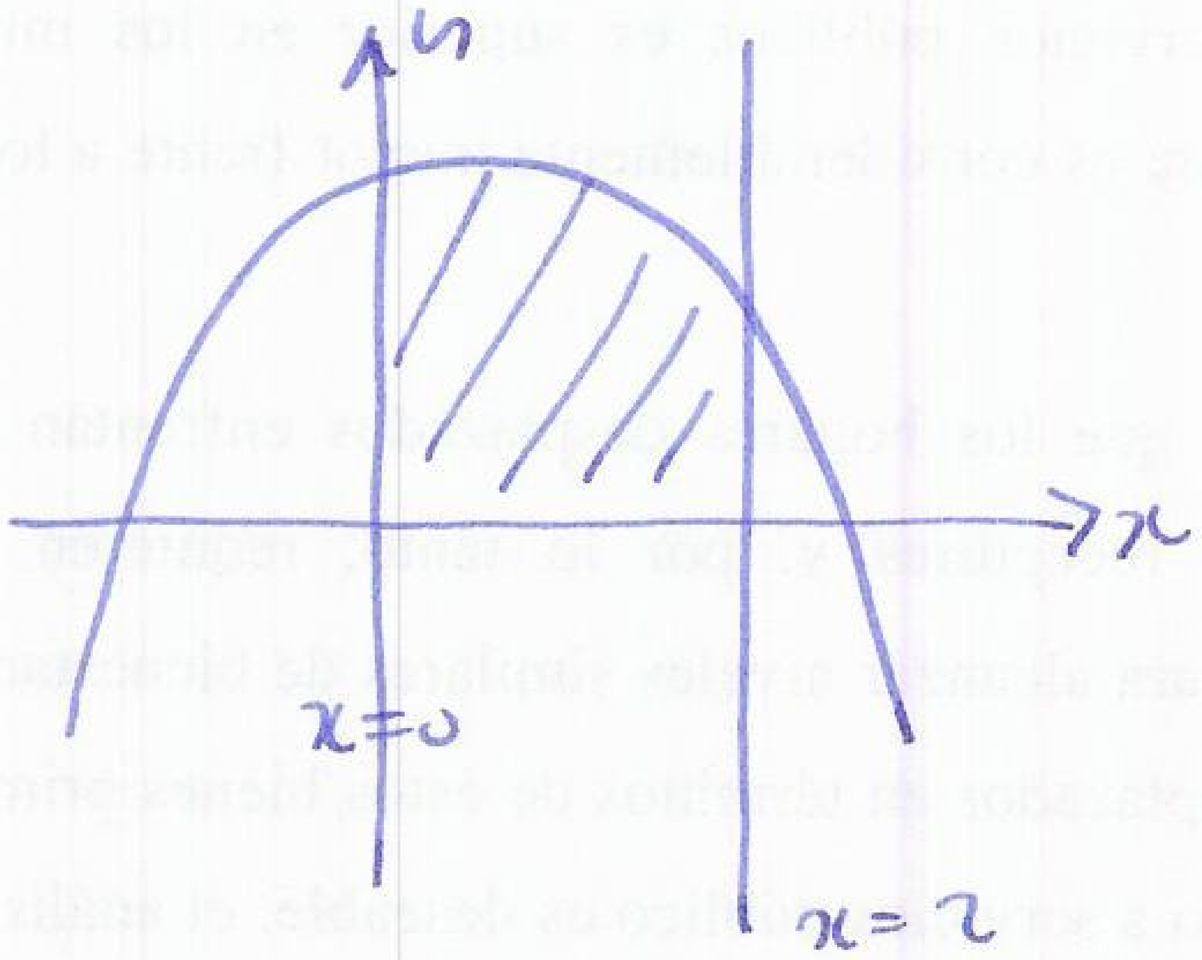
$$y - y_0 = m(x - x_0)$$

$$y - \frac{1}{3} = -\frac{1}{18}(x-2)$$

(c) On your graph above, sketch and label the tangent line at $x = 2$.

(7) (10 points) A region in the plane is bounded by the x-axis, the graph $y = 9 - x^2$, and the lines $x = 0$ and $x = 2$.

(a) Sketch and label the boundaries, and shade in the region.



(b) Compute the area of the region by writing down an integral and evaluating it.

$$\int_0^2 9-x^2\,dx = \left[9x-\tfrac{1}{3}x^3\right]_0^2 = 18-\frac{8}{3} = 15\tfrac{1}{3}.$$

Part II (Do two questions from this part.)

(1) (10 points) Let $f(x) = x^2 - 3x + 1$. Find $f'(x)$ *using the limit definition of the derivative.* Show all your work.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 1 - (x^2 - 3x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h + \cancel{1} - \cancel{x^2} + \cancel{3x} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} 2x + h - 3 = 2x - 3$$

(2) (10 points) The area of a circular oil slick expands at the rate of 20m^2 per minute. How fast is the radius growing when the area is 100m^2?

$A = \pi r^2$

$100 = \pi r^2$

$\Rightarrow r = \frac{10}{\sqrt{\pi}}$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$20 = 2\pi \frac{10}{\sqrt{\pi}} \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{\sqrt{\pi}}$ m/minute

(3) (10 points) Use linear approximation to estimate $\sqrt{17}$. Compare your answer with value you obtain from your calculator.

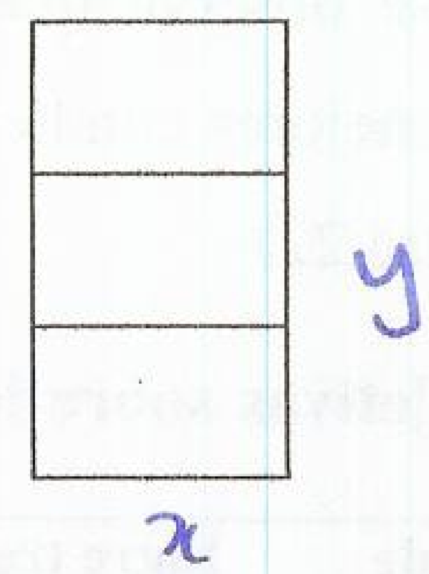
$f(x) = \sqrt{x}$ $\quad f(16) = 4$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(16) = \frac{1}{8}$

$f(16+1) \approx f(16) + f'(16)\cdot 1 = 4 + \frac{1}{8} = 4.125$

actual value $\sqrt{17} \simeq 4.1231$

(4) (10 points) You wish to build a rectangular window frame in the following shape.



The total length of the frame should be 60ft. The total length is the perimeter plus the two horizontal pieces. Determine the width and height which gives the largest area.

$A = xy \qquad P = 4x + 2y = 60$

$$y = 30 - 2x$$

$$A = x(30-2x) = 30x - 2x^2$$

$$\frac{dA}{dx} = 30 - 4x = 0 \Rightarrow x = \frac{15}{2}, \; y = 15$$

(5) (10 points) Use implicit differentiation to find the tangent line to the ellipse $x^2 + 4y^2 = 8$ at the point $(2, 1)$.

$2x + 8y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{x}{4y} = -\frac{1}{2}$ at $(2,1)$

$y - 1 = -\frac{1}{2}(x-2)$