Sample final solutions

Q1 a) $\lim_{x\to 3} \frac{x-3}{x^2-9} = \lim_{x\to 3} \frac{x-3}{(x+3)(x-3)} = \lim_{x\to 3} \frac{1}{x+3} = \frac{1}{6}$

b) $\lim_{x\to 0} \frac{\sin 3x}{x} \underset{\text{l'Hopital}}{=} \lim_{x\to 0} \frac{3\cos 3x}{1} = 3$

c) $\lim_{x\to\infty} \frac{x^2+5x-7}{5x^2+100x+3} = \lim_{x\to\infty} \frac{1+5/x-7/x^2}{5+20/x+3/5x} = \frac{1}{5}$

d) $\lim_{x\to\infty} \frac{x^2+5}{e^{2x}} \underset{\text{l'Hopital}}{=} \lim_{x\to\infty} \frac{2x}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0$

Q2 $\lim_{x\to 0-} 5\cos x - 3 = 2 \qquad \lim_{x\to 0+} 4x+b = b \qquad$ choose $b=2$

Q3 a) $4x^3 + 4e^x + \sec^2(x)$

b) $\frac{1}{2}x^{-1/2} + \frac{(x^2+3) - 2x.x}{(x^2+3)^2}$

c) $\frac{1}{\sin x} \cdot \cos x$

d) $\ln(x) - 2 + x.\frac{1}{x}$

Q4 $(x^2+x+1)y^2 = 3 \qquad (2x+1)y^2 + (x^2+x+1)2yy' = 0$

at $(1,1)$ $\quad 3 + 6y' = 0 \Rightarrow y' = -\frac{1}{2}$

tangent line: $\quad y - y_0 = m(x-x_0) : \quad y - 1 = -\frac{1}{2}(x-1)$

Q5 $g(x) = \frac{1}{4}x^4 - x^2 - 2$

a) $g'(x) = x^3 - 2x \qquad$ critical points: solve $g'(x) = 0$

$x(x^2-2) = 0 \qquad x = 0, \pm\sqrt{2}$

b) $-\quad + \quad - \quad +$ $g'(x)$

$-\sqrt{2} \qquad 0 \qquad \sqrt{2}$

increasing $(-\sqrt{2},0) \cup (\sqrt{2},\infty)$

decreasing $(-\infty,-\sqrt{2}) \cup (0,\sqrt{2})$

c) local max $(0,-2)$ local min $(-\sqrt{2},-3)$ and $(\sqrt{2},-3)$

d) $g''(x) = 3x^2 - 2$ points of inflection $x = \pm\sqrt{2/3}$

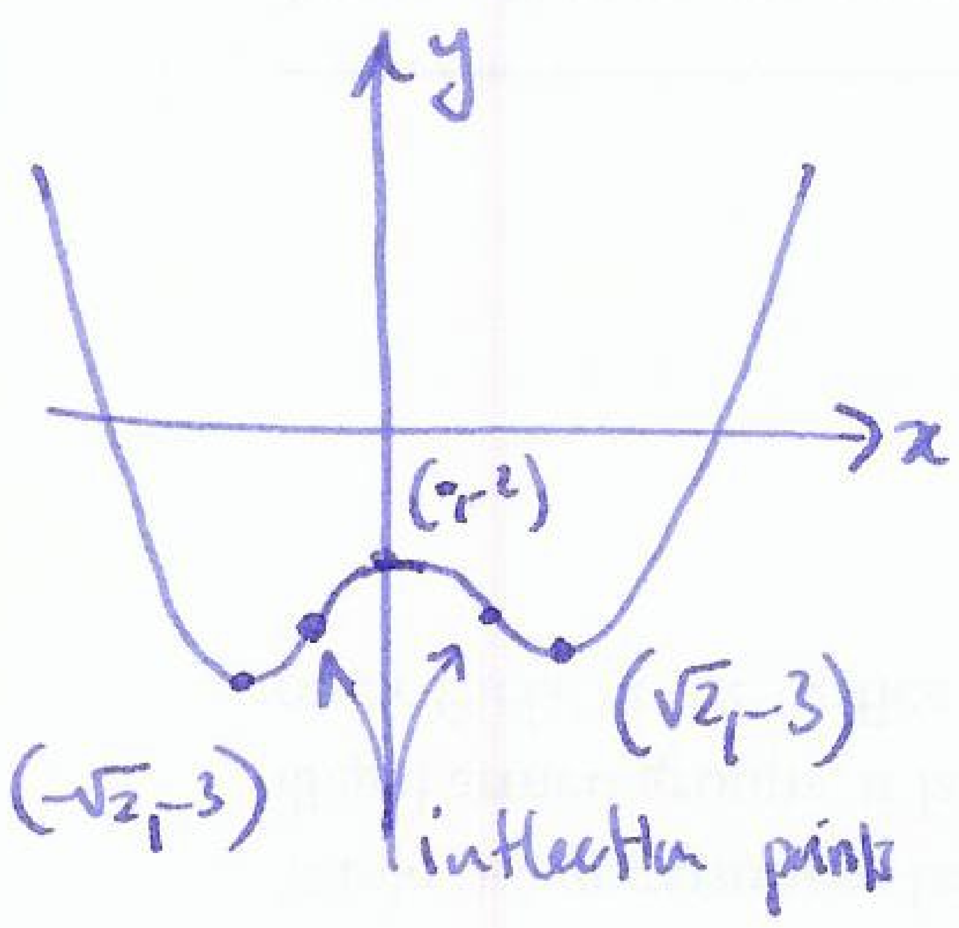
$+ \quad - \quad +$ $g''(x)$

$-\sqrt{2/3} \qquad +\sqrt{2/3}$

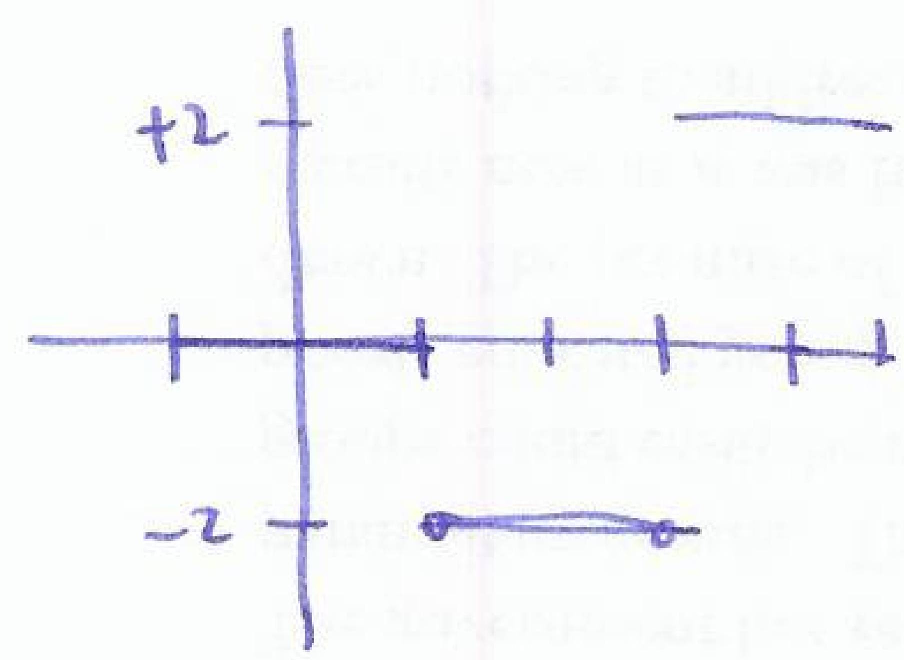
concave up $(-\infty,-\sqrt{2/3}) \cup (\sqrt{2/3},\infty)$

concave down $(-\sqrt{2/3},\sqrt{2/3})$.

e)



Q6 a)



b) 4

Q7

a) $\left[\frac{2x^{3/2}}{3} + \frac{1}{3}x^3 + \ln|x|\right]_1^2 = \frac{2\sqrt{8}}{3} + \frac{8}{3} + \ln 2 - \frac{2}{3} - \frac{1}{3}$

b) $e^x - e^{-x} + 2x + C$

c) $\left[-\frac{1}{3}\cos(3x)\right]_0^{\pi/6} = -\frac{1}{3}\cos\left(\frac{\pi}{2}\right) + \frac{1}{3}\cos(0) = \frac{1}{3}$

d) $\left[\frac{1}{4}e^{x^4}\right]_0^2 = \frac{1}{4}(e^{16} - 1)$

Q8 a) $a(t) = x'' = -32$

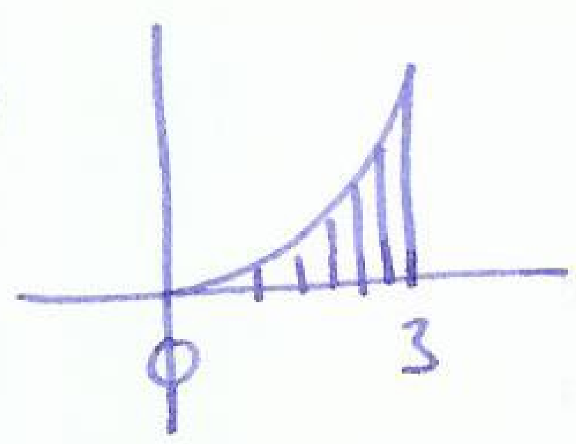
$v(t) = x' = -32t + c \qquad v(0) = 128 \Rightarrow c = 128$

$x(t) = -16t^2 + 128t + c \qquad x(0) = 0 \Rightarrow c = 0$

max height when $v = 0$, i.e. $t = \frac{128}{32} = 4 \qquad x(4) = -16.16 + 128.4 = 256$

b) i) $\int_0^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^3 = 9$

ii)



$L_n = \sum_{k=0}^{n-1} f\left(\frac{3k}{n}\right)\frac{3}{n} = \sum_{k=0}^{n-1} \frac{9k^2}{n^2}\cdot\frac{3}{n} = \frac{27}{n^3}\sum_{k=0}^{n-1} k^2$

$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{27}{n^3}\frac{(n-1)n(2n-1)}{6} = \lim_{n\to\infty} 9\frac{(n-1)(n-1/2)}{n^2} = \lim_{n\to\infty} 9(1-1/n)(1-1/2n) = 9$

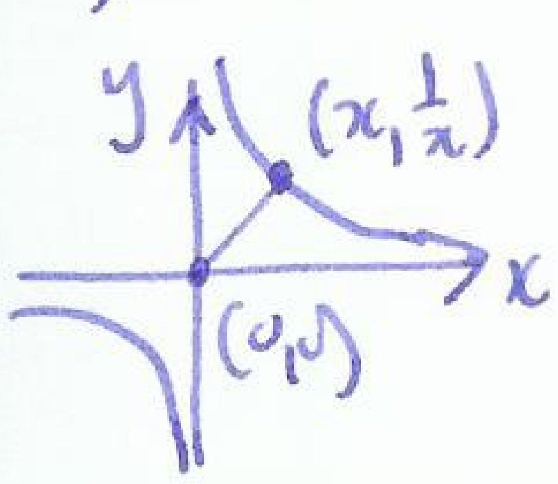
c) $f(x) = \sqrt{x} \qquad f(100) = 10 \qquad f(100+3) \simeq f(100) + f'(100).3$

$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(100) = \frac{1}{20} \qquad \simeq 10 + \frac{3}{20} = 10.15$

actual value: $\sqrt{103} \simeq 10.1489$ (close to 10.15)

d) $d^2 = x^2 + \frac{1}{x^2} = f(x). \qquad f'(x) = 2x - 2x^{-3} = 0$

$x^4 = 1 \qquad x = \pm 1$ (want +ve soln).



closest point is (1,1).