MTH 700 Topology I, Fall 12, HW2

(1) Let X be a topological space, and define relation $x \sim y$ if x and y lie in a common connected component. Show that $\sim$ is an equivalence relation. Show that if $\{A_i\}_{i \in I}$ is a collection of connected subsets of X with $A_i \cap A_j \neq \emptyset$ for all i and j, then the union of the A_i is connected.

(2) Prove that if X has only finitely many connected components, then all the components are open.

(3) Show that a contractible space is path connected.

(4) Show that a connected open subset of $\mathbb{R}^2$ is path connected.

(5) Prove that X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

(6) A retract A of a space X is a subset $A \subset X$ such that there is a continuous map $r \colon X \to A$ such that $r|_A$ is the identity on A. Show that every retract A of a Hausdorff space X is closed in X.

(7) Show that composition of paths satisfies the following property. If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$, and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

(8) Let h be a path from x_0 to x_1. Show that the change of basepoint map $\beta_h \colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ only depends on the homotopy class of h.