MTH 700 Topology I, Fall 12, HW1

(1) Let (X,d) be a metric space, and let $r < s \in \mathbb{R}$. Show that $\overline{B(x,r)} \subset B(x,s)$.

(2) In a topological space X show that
   (a) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, and equality need not hold.
   (b) $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$, and equality need not hold.

(3) Let ℓ^2 denote the set of square summable sequences, i.e. $\ell^2 = \{(a_1, a_2, a_3, \dots) \mid \sum a_i^2 < \infty\}$. The following is a metric on ℓ^2 (you don't have to prove this):

$$d((a_1, a_2, \dots), (b_1, b_2, \dots)) = \sqrt{\sum (a_i - b_i)^2}.$$

Show that the unit ball $B = \{(a_1, a_2, a_3, \dots) \mid \sum a_i^2 \leq 1\}$ is not compact. (Hint: show B is not sequentially compact).

(4) A topological space is called *separable* if it contains a countable dense subset. A topological space is called *Lindelof* if every open cover contains a countable subcover. A topological space is called *second countable* if it has a countable base for the topology. Prove:
   (a) A space that is second countable is separable.
   (b) A space that is second countable is Lindelof.
   (c) A separable metric space is second countable.

(5) Let $(\mathbb{R}, T)$ be the real line with the following topology: A base consists of the set of half-open intervals $[a,b)$. The space $(\mathbb{R}, T)$ is called the lower limit topology on $\mathbb{R}$.
   (a) Show that $(\mathbb{R}, T)$ is not connected.
   (b) Show that $(\mathbb{R}, T)$ is separable but not second countable, hence not metric.
   (c) Show that $(\mathbb{R}, T)$ is Lindelof.
   (d) Show that $(\mathbb{R}, T) \times (\mathbb{R}, T)$ is separable but not Lindelof.

(6) Show that $(0,1)$ and $[0,1]$ are not homeomorphic.

(7) Show that there is no continous injective map $\mathbb{R}^2 \to \mathbb{R}$, and hence these two spaces are not homeomorphic.