Math 232 Calculus 2 Fall 12 Midterm 2a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no phones.
- You may use a 3×5 inch index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) Find $\int_0^{\pi/4} \sin^5 x \, dx.$ $= \int_0^{\pi/4} \sin x \, (1-\cos^2 x)^2 \, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x.$

$= \int_1^{\sqrt{2}/2} \sin x \, (1-u^2)^2 \frac{1}{-\sin x} \, du$

$= -\int_1^{\sqrt{2}/2} 1 - 2u^2 + u^4 \, du = -\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right]_1^{\sqrt{2}/2}$

$= -\left(\frac{\sqrt{2}}{2} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5 \right) + \left(1 - \frac{2}{3} + \frac{1}{5} \right)$

(2) Find $\int \sin(4x)\cos(7x)\,dx$.

$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$

$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$

$\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$.

$\frac{1}{2}\int \sin(11x) + \sin(-3x)\,dx = \frac{1}{2}\left(-\frac{1}{11}\cos(11x) + \frac{1}{3}\cos(-3x)\right) + C$

$\cos^2 u + \sin^2 u = 1$

$\cos^2 u = 1 - \sin^2 u$

(3) Find $\int \sqrt{4 - x^2}\, dx$.

$x = 2 \sin u$

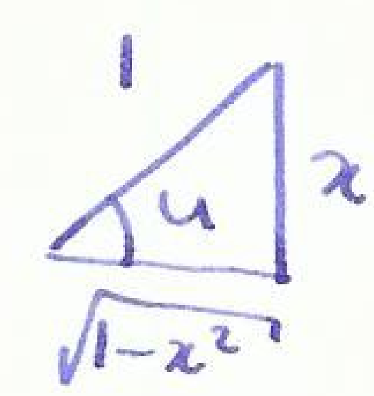
$\frac{dx}{du} = 2\cos u$

$\cos 2u = \cos^2 u - \sin^2 u$

$= 2\cos^2 u - 1$.

$= \int \sqrt{4 - 4\sin^2 u}\; 2\cos u \, du = 4 \int \cos^2 u \, du$

$= 4 \int \frac{1}{2}\cos 2u + \frac{1}{2} \, du = 2\left[\frac{1}{2}\sin 2u + u\right] = \sin 2u + 2u + c$



$= 2\sin u \cos u + 2\sin^{-1}\left(\frac{x}{2}\right) + c$

$= 2x\sqrt{1-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + c$.

(4) Find $\int \frac{3}{x^2-4}\,dx$.

$$\frac{3}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$x=2: \quad 3 = 4A$

$x=-2: \quad 3 = -4B$

$A = 3/4 \quad B = -3/4.$

$$\int \frac{3/4}{x-2} - \frac{3/4}{x+2}\,dx = \frac{3}{4}\ln|x-2| - \frac{3}{4}\ln|x+2| + c$$

$u = x \qquad u' = 1$

$v' = e^{-2x} \qquad v = -\frac{1}{2}e^{-2x}$

(5) Find $\int_1^\infty \underbrace{x}_{u}\underbrace{e^{-2x}}_{v'}\,dx$.

$$= \lim_{R\to\infty}\int_1^R xe^{-2x}\,dx = \lim_{R\to\infty}\left[-\frac{1}{2}xe^{-2x}\right]_1^R + \int_1^R \frac{1}{2}\cancel{x}e^{-2x}\,dx$$

$$= \lim_{R\to\infty} \underbrace{-\frac{1}{2}Re^{-2R}}_{\to 0} + \frac{1}{2}e^{-2} + \left[-\frac{1}{4}e^{-2x}\right]_1^R$$

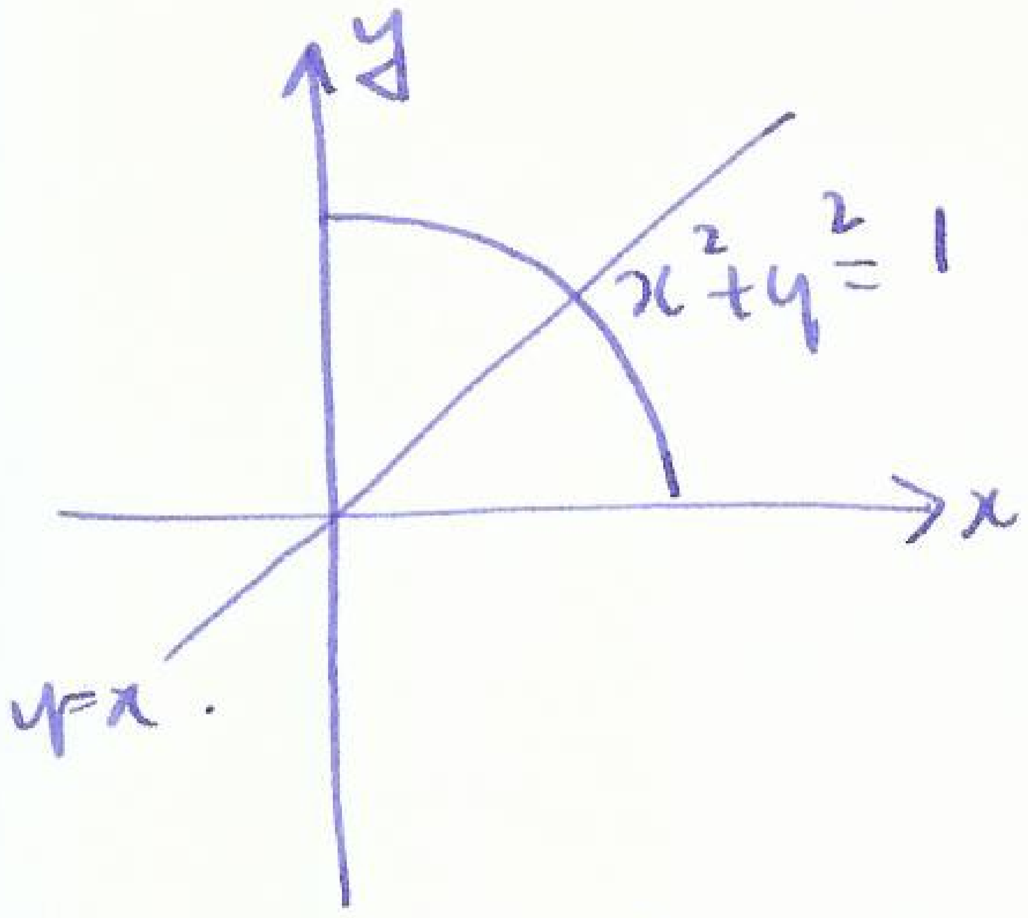
$$= \lim_{R\to\infty} \frac{1}{2}e^{-2} \underbrace{-\frac{1}{4}e^{-2R}}_{\to 0} + \frac{1}{4}e^{-2} = \frac{1}{2}e^{-2} + \frac{1}{4}e^{-2} = \frac{3}{4}e^{-2}.$$

(6) Find $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$.

$$\lim_{R\to 0} \int_R^1 x^{-1/2}\, dx = \lim_{R\to 0} \left[2x^{1/2} \right]_R^1 = \lim_{R\to 0} \left(2 - 2\sqrt{R} \right) = 2 .$$

(7) Find the center of mass of the region with constant density $\rho = 1$ inside the circle $x^2 + y^2 = 1$, and in the first quadrant, i.e. $x \geq 0$ and $y \geq 0$.

Hint: You may use the fact that the area of the whole disc is πr^2, and, by symmetry, you may assume that the center of mass lies on the line $y = x$.



area of circle $= \pi$

area of quadrant $= \frac{\pi}{4}$

$$M_x = \int_0^1 x\sqrt{1-x^2}\,dx$$

sub $u = 1-x^2$, $\frac{du}{dx} = -2x$

$$M_x = \int_1^0 x\,u^{1/2}\frac{dx}{du}\,du = \int_1^0 x \cdot u^{1/2}\frac{1}{-2x}\,dx = \frac{1}{2}\int_0^1 u^{1/2}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3} \qquad \bar{x} = \frac{M_x}{M} = \frac{1/3}{\pi/4} = \frac{4}{3\pi}$$

center of mass $(\bar{x}, \bar{y}) = \left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

(8) Find the degree three Taylor polynomial for the function $f(x) = \sin(x^2)$.

$f(x) = \sin(x^2)$ $\qquad$ $f(0) = 0$

$f'(x) = \cos(x^2).2x$ $\qquad$ $f'(0) = 0$

$f''(x) = -\sin(x^2).4x^2 + 2\cos(x^2)$ $\qquad$ $f''(0) = 2$

$f^{(3)}(x) = -\cos(x^2).8x^3 - \sin(x^2).8x + -2\sin(x^2).2x.$ $\qquad$ $f'''(0) = 0$

$T_3(x) = \frac{2x^2}{2} = x^2.$

(9) Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2+n}$ converge or diverge? Justify your answer.

Comparison test: $\frac{1}{n^2+n} < \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p<1$ or integral test)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

Integral test: $\int_1^{\infty} \frac{1}{x^2+x}\,dx = \int_1^{\infty} \frac{1}{x(x+1)}\,dx$ $\qquad \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$= \frac{A(x+1)+Bx}{x(x+1)}$

$x=0: \; A=1$

$x=-1: \; B=-1$

$$= \int_1^{\infty} \frac{1}{x} - \frac{1}{x+1}\,dx$$

$$= \Big[\ln|x| - \ln|x+1|\Big]_1^{\infty}$$

$$= \lim_{R\to\infty} \ln\left|\frac{R}{R+1}\right| + \ln(2) = \ln(2) + \lim_{R\to\infty} \ln\left|\frac{1}{1+1/R}\right| = \ln(2)$$

converges.

(10) Does the series $\sum_{n=2}^{\infty} \frac{n}{4^n}$ converge or diverge? Justify your answer.

Integral test: $\int_0^{\infty} \underbrace{x}_{u} \underbrace{e^{-\ln(4)x}}_{v'} \, dx = \left[x \cdot -\frac{1}{\ln(4)} e^{-\ln(4)x} \right]_0^{\infty} + \int_0^{\infty} \frac{1}{\ln(4)} e^{-\ln(4)x} \, dx$

$u = x \quad u' = 1$

$v' = e^{-\ln(4)x} \quad v = -\frac{1}{\ln(4)} e^{-\ln(4)x}$

$= \left[\frac{-1}{\ln(4)^2} e^{-\ln(4)x} \right]_0^{\infty} = \frac{1}{\ln(4)}.$

$\Rightarrow \sum_{n=2}^{\infty} \frac{n}{4^n}$ converges.

comparison test: $n \leq 2^n$ so $\frac{n}{4^n} \leq \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges as geometric series $\Rightarrow \sum_{n=2}^{\infty} \frac{n}{4^n}$ converges.