Math 329 Geometry Spring 11 Midterm 2

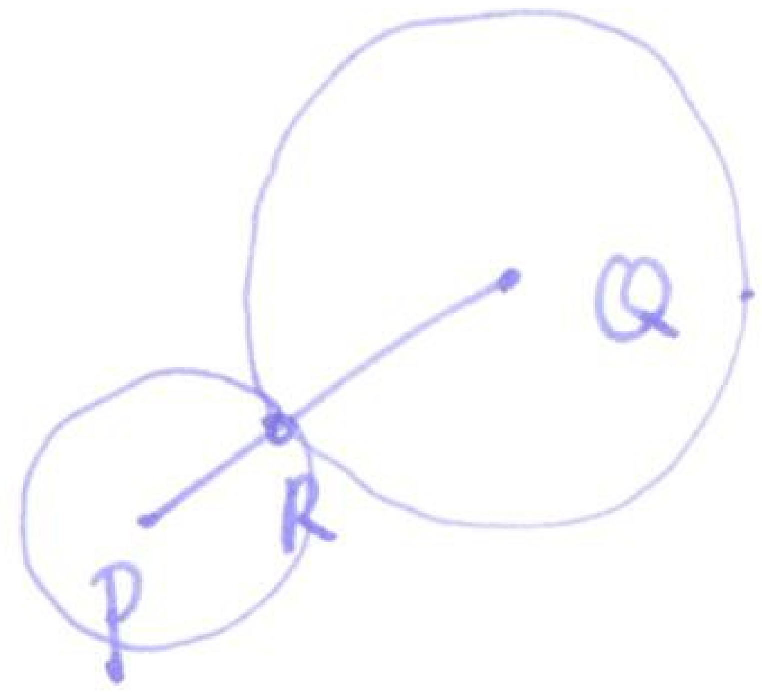
Name: Solutions

- You may use a compass and straight edge, but no notes.
- Do any six questions fromthe following eight questions.

1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
	120	

Midterm 2	
Overall	

(1) (20 points) Let f be any isometry of $\mathbb{R}^2$, which fixes two points P and Q, i.e. $f(P) = P$ and $f(Q) = Q$. Show that f fixes every point on the line connecting P and Q.



$$d(f(R), f(P)) = d(R, P) = d(f(R), P).$$

$$d(f(R), f(Q)) = d(R, Q) = d(f(R), Q)$$

so $f(R)$ lies on intersection of circle of radius $d(R,P)$ about P and circle of radius $d(R,Q)$ about Q.

but these circles intersect exactly once at R, so $d(P,Q) = d(P,R) + d(R,Q)$

so $f(R) = R$.

so f fixes every point on the line segment connecting P and Q.

Alternate method: by classification of isometries, f is one of:

rotation : exactly one fixed point

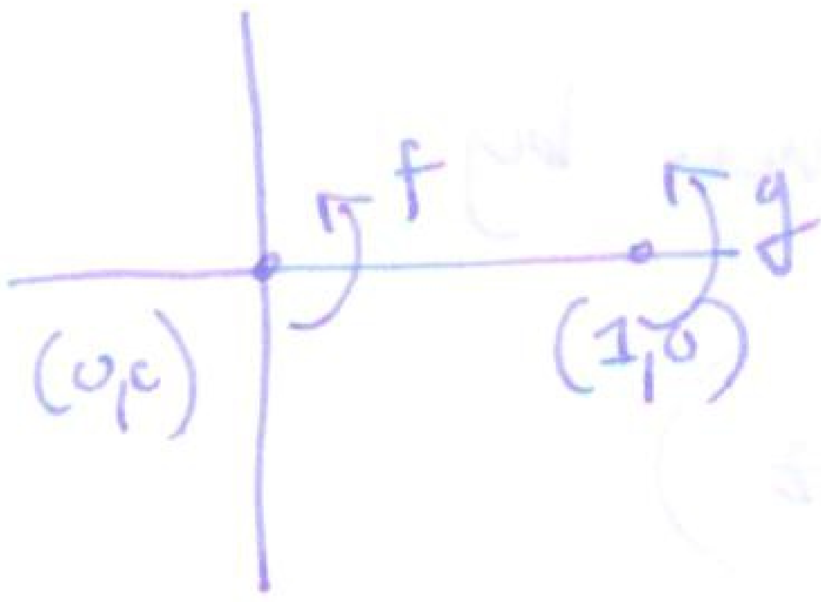
translation : no fixed point

reflection : fixed line

glide reflection : no fixed point.

so f must be a reflection, and P, Q must lie on the fixed line, so entire line containing PQ fixed

(2) (20 points) Show that the product of a rotation by π about $(0,0)$ followed by a rotation by π about $(1,0)$ is a translation. Find an explicit description of the translation.



$f: z \mapsto -z$

$g: z \mapsto z-1 \mapsto -(z-1) = -z+1$
$\mapsto -z+2$

$g \circ f(z) = -(-z)+2 = z+2$

so $g \circ f$ is the translation $z \mapsto z+2$

(3) (20 points) Describe an isometry of S^2 which preserves the equator, and swaps the north and south poles (i.e. ~~$(1,0,0)$~~ and ~~$(-1,0,0)$~~). (There's more than one, just describe a particular one.)

$(0,0,1)$ $(0,0,-1)$

Reflection in equator, given by

$$(x,y,z) \mapsto (x,y,-z)$$

sends $(0,0,1) \mapsto (0,0,-1)$

and $(x,y,0) \mapsto (x,y,0)$ so preserves equator.

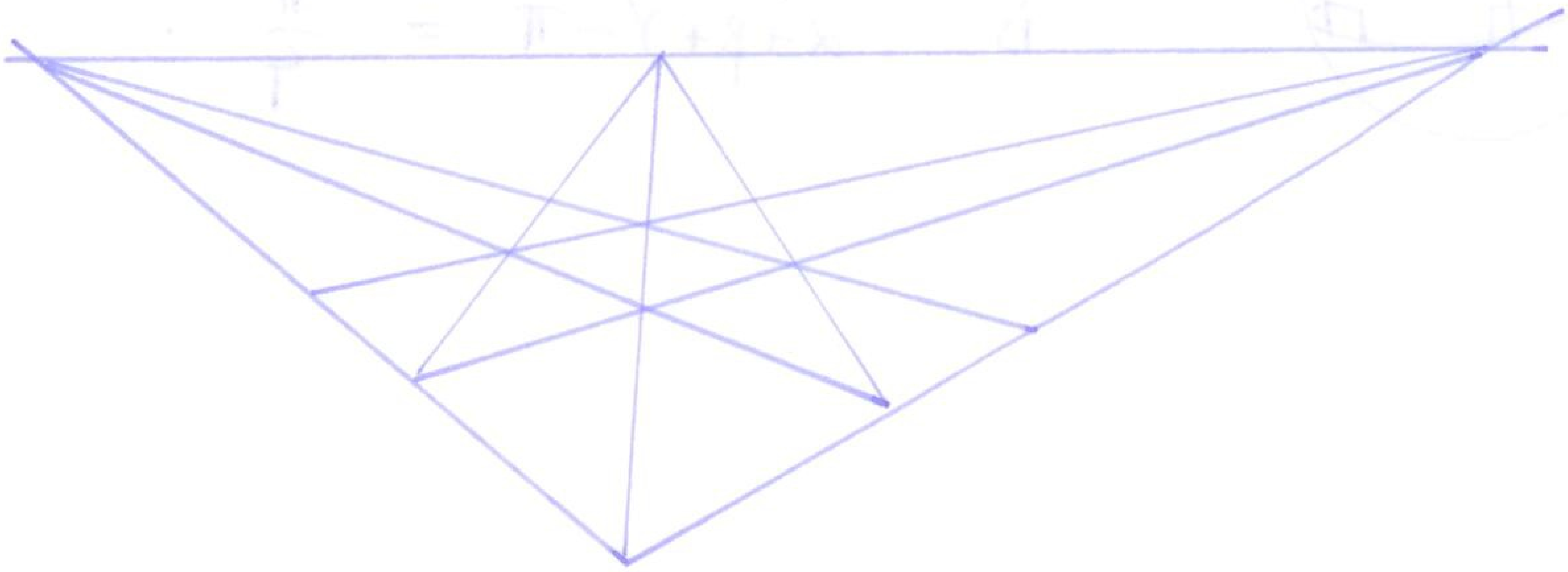
antipodal map $(x,y,z) \mapsto (-x,-y,z)$ also works.

(4) Is there a triangle on the unit sphere with two right angles and an angle of $\pi/4$? If so draw a picture of it and find its area.

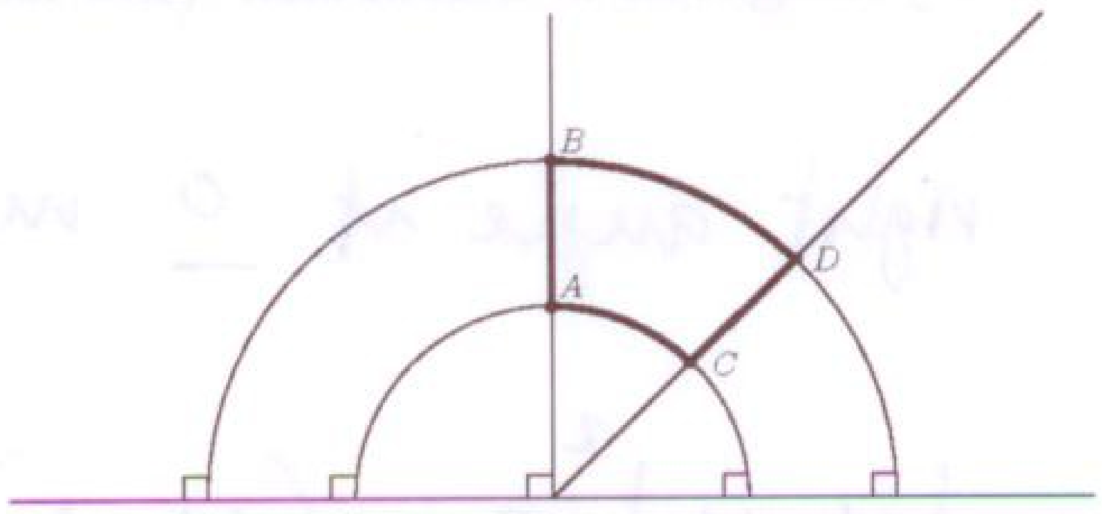
area of spherical triangle

is $\alpha+\beta+\gamma-\pi = \frac{\pi}{4}$.

(5) (20 points) Use only a straight edge to construct a perspective drawing of a tiled floor. Draw at least four tiles.



(7) (20 points) Consider the curves AB, BD, CD and AC drawn in the upper halfspace model for the hyperbolic plane below.



(a) Which of the curves are hyperbolic straight lines, if any?

(b) Which of the curves have endpoints the same distance apart, if any? Justify.

must

AB, BD, AC hyperbolic lines

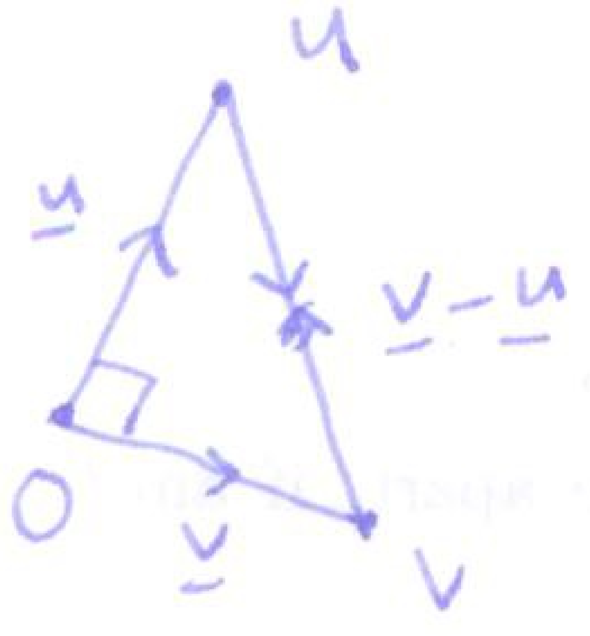
CD not hyperbolic line.

AC, BD same length

CD longer than AB.

CD, AB may be different from BD

(6) (20 points) Let $\mathbf{0}, \mathbf{u}$ and $\mathbf{v}$ be the vertices of a right angled triangle. Use vectors to prove Pythagoras's Theorem. (Do not use the law of cosines.)



right angle at $\underline{0}$ means $\underline{u}.\underline{v} = 0$

$$|\underline{v}-\underline{u}|^2 = (\underline{v}-\underline{u}).(\underline{v}-\underline{u})$$

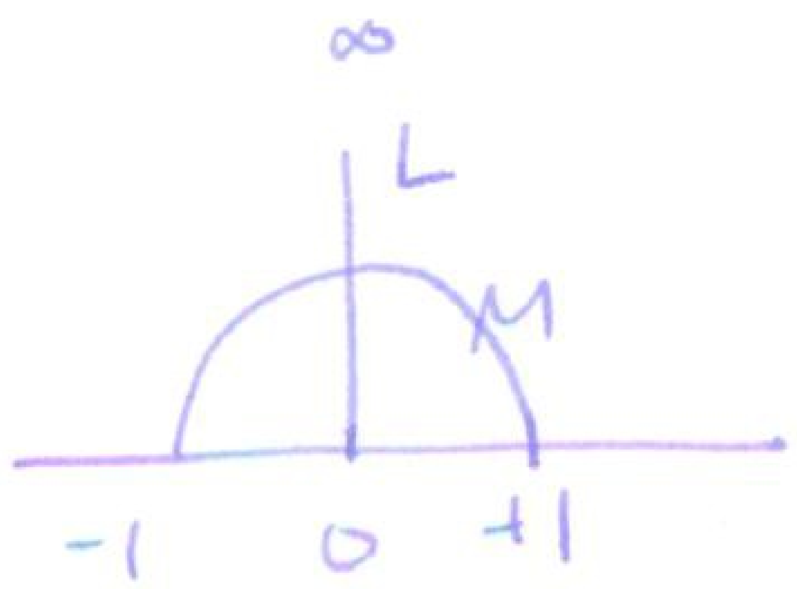
$$= \underline{v}.\underline{v} \underbrace{- \underline{u}.\underline{v} - \underline{v}.\underline{u}}_{\underline{u}.\underline{v} = 0\,.} + \underline{u}.\underline{u}$$

$$= |\underline{v}|^2 + |\underline{u}|^2$$

(8) (20 points) Let L be the y-axis in the upper halfspace model, and let M be the hyperbolic line with endpoints $+1$ and -1.

(a) Write down an explicit hyperbolic isometry that takes M to L.

(b) Where does the point i go to under your isometry? Are there isometries which take i to other points on L, and if so which points can you send i to?



a) $$z \mapsto -\frac{z+1}{z-1} = \frac{-z-1}{z-1}$$

b) $$i \mapsto -\frac{i+1}{i-1}\cdot\frac{-i-1}{-i-1} = -\frac{1-i-i-1}{1+1} = i$$

can post compose with any isometry $z \mapsto kz$ $k \in \mathbb{R}$ which fixes L, so can send $M \to L$ with i going to any point on L.