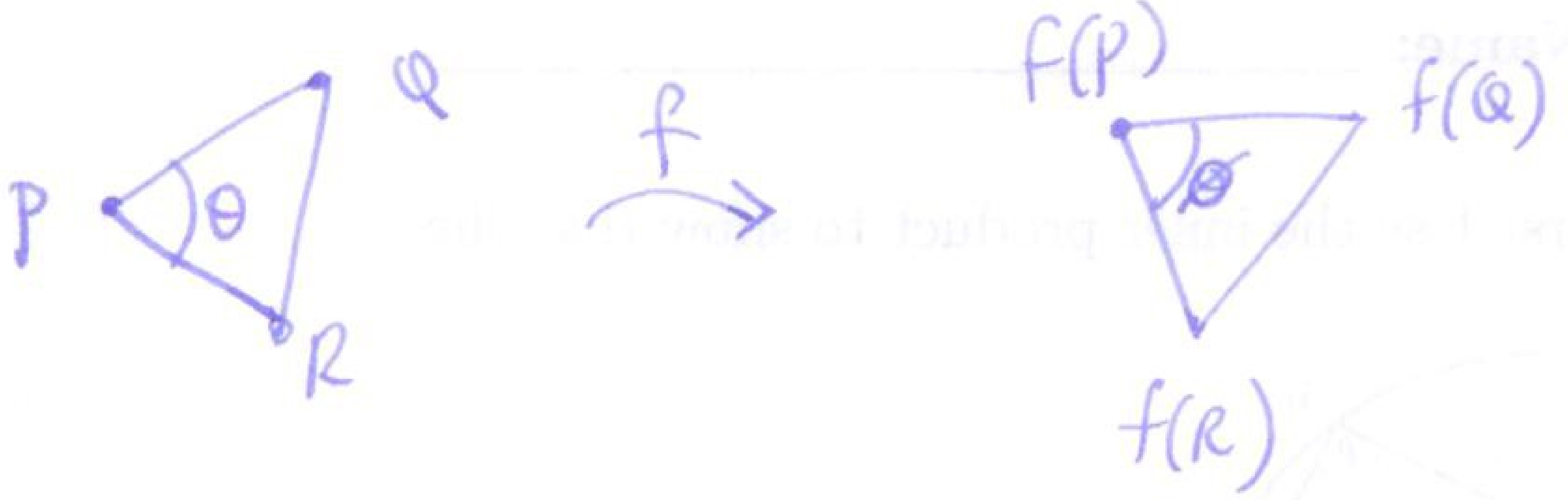
Geometry Spring 11 Sample midterm 2 — Solutions



**Q1** $f:\mathbb{R}^2\to\mathbb{R}^2$, preserves distances.



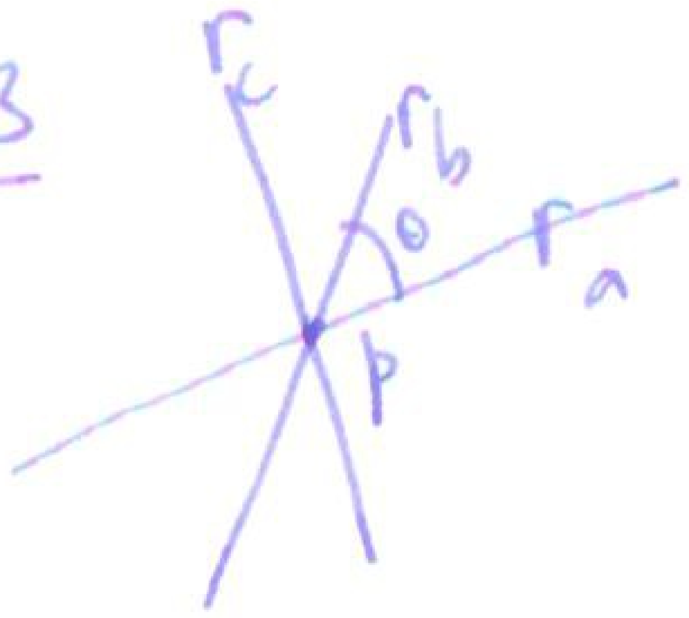

$\Delta f(P)f(Q)f(R)$ is congruent to $\Delta PQR$ (same side lengths) so $\theta=\phi$, so $f$ preserves angles.

**Q2**


choose a triangle disjoint from $P$ with three right angles. This gets maps to three intersecting lines which form a triangle,


with total interior angle $\pi$, so $f$ can't preserve angles.

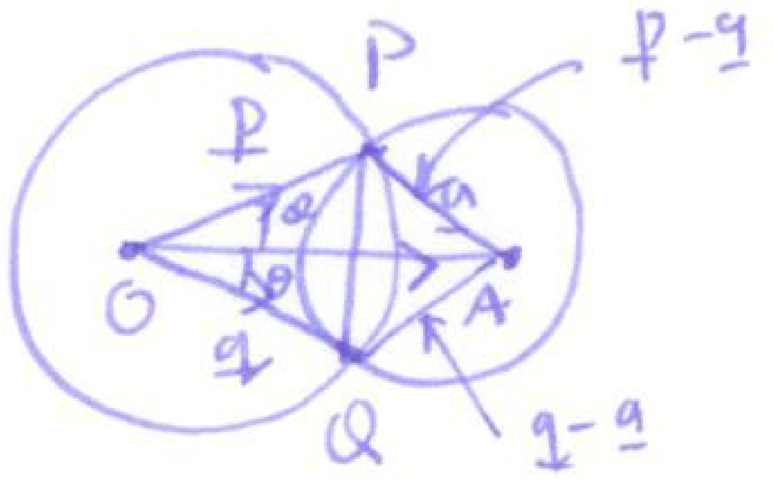
**Q3**




$r_b r_a$ is rotation about $P$ by angle $2\theta$.

so $r_c r_b r_a$ is rotation about $P$ followed by reflection in a line through $P$, so in fact is reflection in a line at angle $\theta$ clockwise to $c$. $[z\mapsto e^{2i\theta}z\mapsto e^{-2i\theta}\bar{z}$, reflection in line at angle $-\theta]$

**Q4**




$\underline{a}\cdot(\underline{q}-\underline{p}) = \underline{a}\cdot\underline{q}-\underline{a}\cdot\underline{p}$ want this to be zero.

note: $|\underline{p}|=|\underline{q}|$ and $|\underline{p}-\underline{a}|=|\underline{q}-\underline{a}| \Rightarrow (\underline{p}-\underline{a})\cdot(\underline{p}-\underline{a})=(\underline{q}-\underline{a})\cdot(\underline{q}-\underline{a})$

$$|p|^2 - 2g\cdot p + |g|^2 = |q|^2 - 2g\cdot q + |g|^2$$

$\Rightarrow$

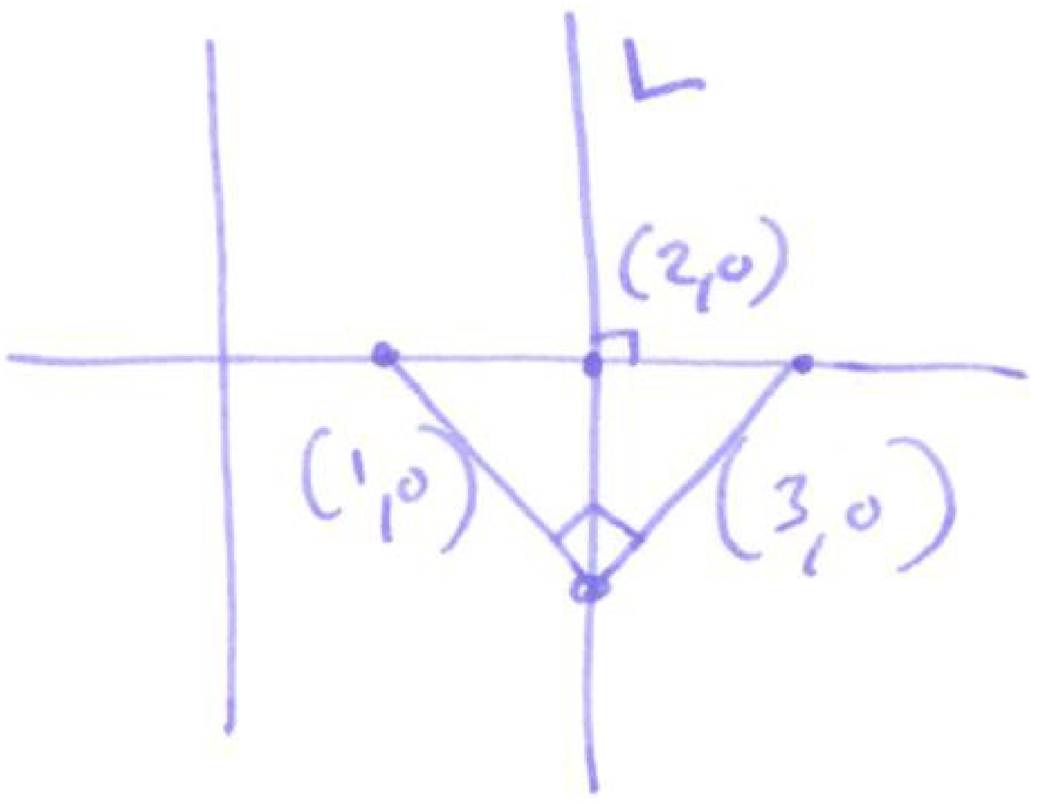
$$g\cdot p = g\cdot q.$$

$\therefore$ $g\cdot(q-p) = g\cdot q - g\cdot p = 0$ as required $\square$.

**Q5** a) $\frac{1}{3}\big((1,0,0)+(0,1,0)+(0,0,1)\big) = (1/3, 1/3, 1/3)$.

b) $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

**Q6**




a) fixed point of rotation is equidistant from any point and its image. equidistant set is perpendicular bisector L

$L \leftrightarrow x = 2$.

b) $(2, -1)$ (~~equilat~~ isosceles right angled triangle with hyp length 2 has other sides length $\sqrt{2}$, and height 1).

**Q7** a) identity
rotations about $(0,0)$
reflections in lines through $(0,0)$. $\Big\}\ (*)$

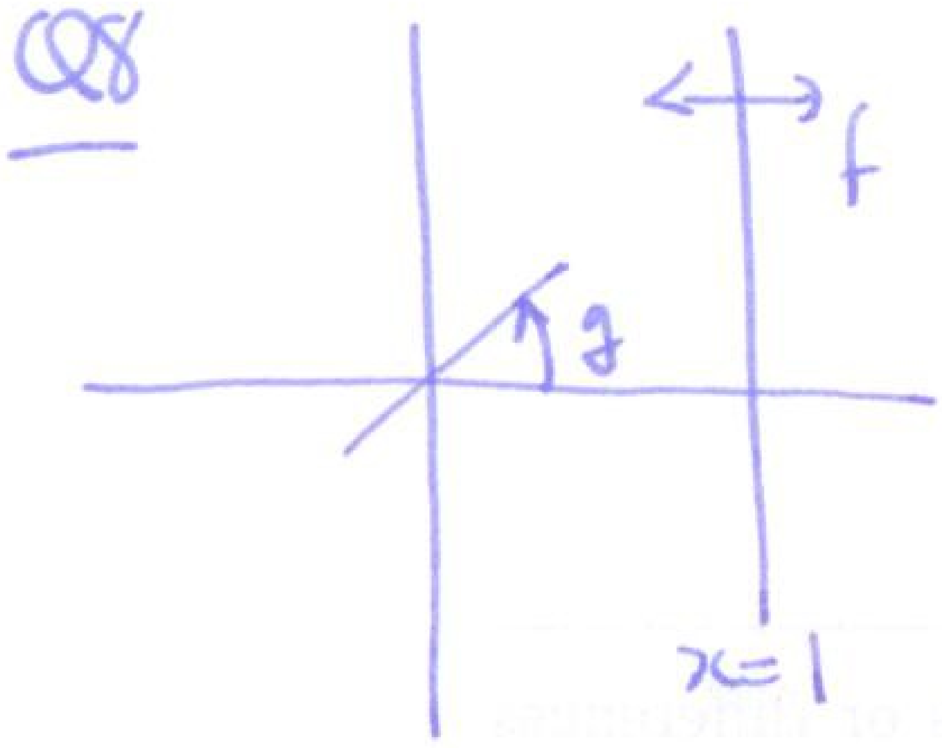
b) Let $t$ be translation $(x,y) \mapsto (x+1, y+1)$

Let $f$ be any isometry which takes $(0,0)$ to $(1,1)$

then $ft^{-1} = h$ say, and $h$ is one of $(*)$ so $f = ht$,

so any isometry which takes $(0,0)$ to $(1,1)$ is the translation $t$, followed by one of the isometries $(*)$.

Q8




$f: z \mapsto z-1 \mapsto -\overline{(z-1)} \mapsto -\bar{z}+2.$

$g: z \mapsto e^{\pi/4 i}$

$$f\circ g(z) = -\overline{(e^{\pi/4 i}z)} + 2 = -e^{-\pi/4 i}\bar{z}+2$$
$$= e^{3\pi/4 i}\bar{z}+2$$

this is of the form $z \mapsto a\bar{z}+b$ so is either a reflection or a glide reflection.

conjugate by $z \mapsto e^{-\frac{3\pi}{8}i}$: (clockwise rotⁿ by $\frac{3\pi}{8}$)

$$z \mapsto e^{3\pi/8 i} \mapsto e^{3\pi/4 i} e^{-\frac{3\pi}{8}i}\bar{z}+2$$
$$= e^{3\pi/8 i}\bar{z}+2$$
$$\mapsto \bar{z} + \underbrace{2e^{-\frac{3\pi i}{8}}}_{} \quad (*)$$

not purely imaginary so this is a glide reflection.

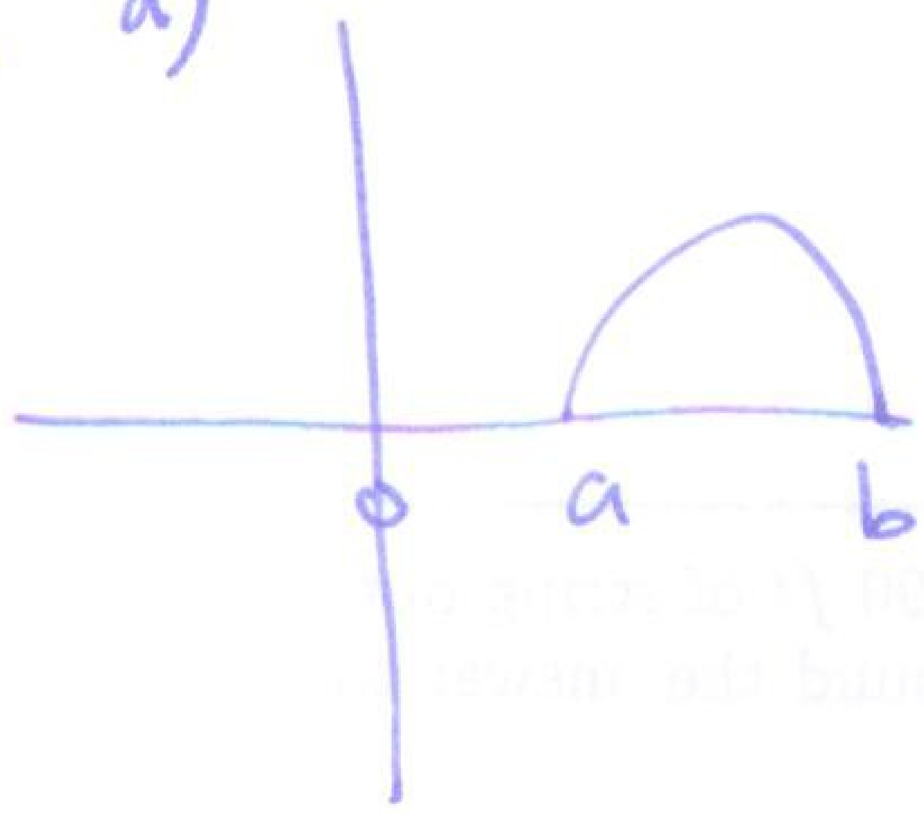
recall: if $z \mapsto \bar{z}+r$ ($r \in \mathbb{R}$) is a glide reflection, then conjugation by $z \mapsto z+is$ ($s \in \mathbb{R}$) gives:

$z \mapsto z - is \mapsto \bar{z}+is+r \mapsto \bar{z}+2is+r$

so $(*)$ is a glide reflection with translation $2\cos(-\frac{3\pi}{8})$ along the line $y = -\sin(-\frac{3\pi}{8}) = \sin(3\pi/8)$

so the original line is $y = \sin(3\pi/8)$ rotated anticlockwise by $3\pi/8$.

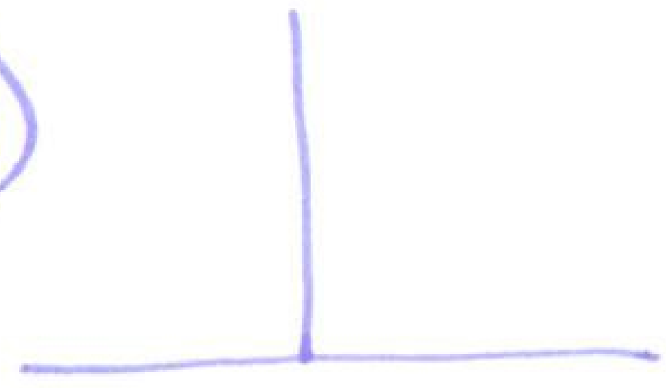
**Q9** a)




$$a \mapsto 0$$
$$b \mapsto \infty$$

$f(z) = \dfrac{z-a}{z-b}$ works.

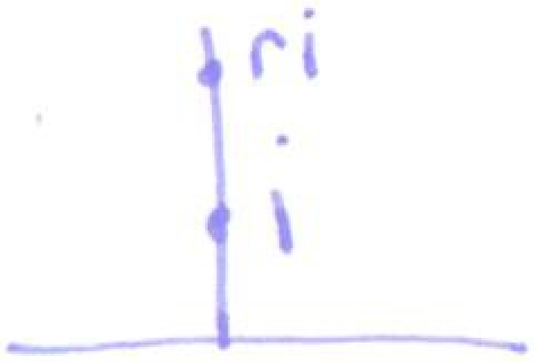
b)



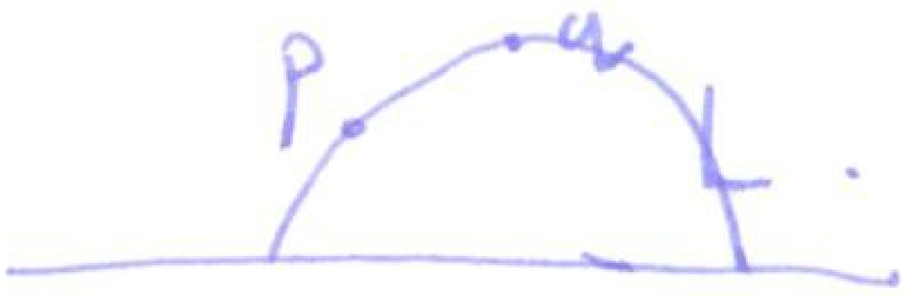
Euclidean dilations $z \mapsto az$, $a \in \mathbb{R}$
(hyperbolic translations)

Reflection/glide reflections in y-axis.

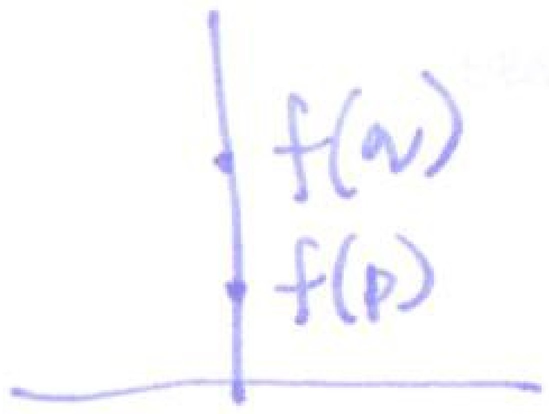
c) Let $i$, $ri$ be two points distance $d$ apart on y-axis.



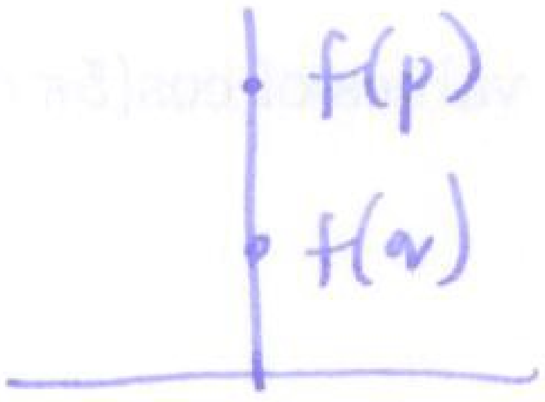

Let $p, q$ be any other points distance $d$ apart, and let $L$ be the line containing them




use a) to take $L$ to y-axis. by $f$ say




or




now use a dilation to take the lower point to $i$, upper point must get mapped to $ri$.

so any pair of points $p, q$ distance $d$ apart ~~is~~ may be taken to $(i, ri)$ so all pairs equivalent.