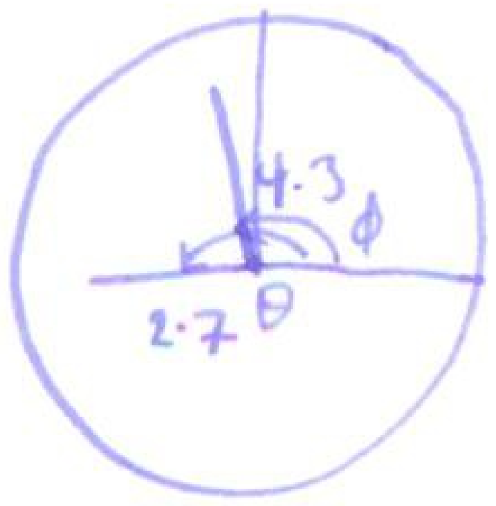
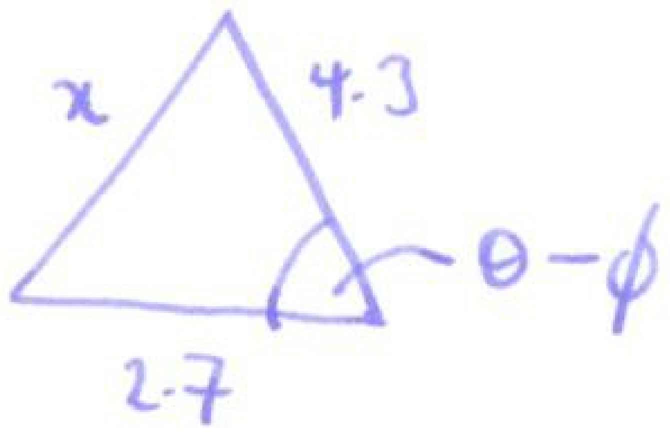
Sample midterm 3 solutions

Q1



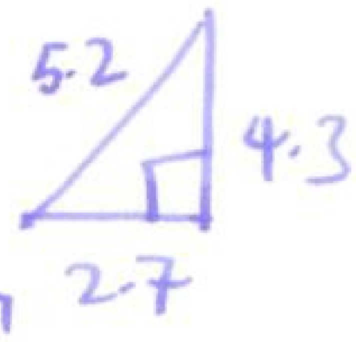
$$\frac{d\phi}{dt} = -2\pi \text{ radians/hour} \qquad \frac{d\theta}{dt} = -\frac{2\pi}{12} \text{ radians/hour.}$$



$$x^2 = (4.3)^2 + (2.7)^2 - 2(4.3)(2.7)\cos(\theta-\phi)$$

$$2x\frac{dx}{dt} = 2(4.3)(2.7)\sin(\theta-\phi)\left(\frac{d\theta}{dt} - \frac{d\phi}{dt}\right)$$

at ~~4 o'clock~~ 9 o'clock



$$5.2 \approx \sqrt{(2.7)^2+(4.3)^2}$$

$$\frac{dx}{dt} = \frac{(4.3)(2.7)}{(5.2)}\sin\left(\frac{\pi}{4}\right)\left(-\frac{\pi}{6}+2\pi\right)$$

$$\approx 9.1 \text{ m/hour.}$$

Q2

$$f(x) = \tan^{-1}(x)$$

$$f'(x) = \frac{1}{1+x^2}$$

$$\tan^{-1}(0.9) \approx \tan^{-1}(1) + f'(1).(-0.1)$$

$$\approx \frac{\pi}{4} + \frac{1}{2}.0.1 \approx \frac{\pi}{4} - 0.05. \approx 0.735$$

Q3

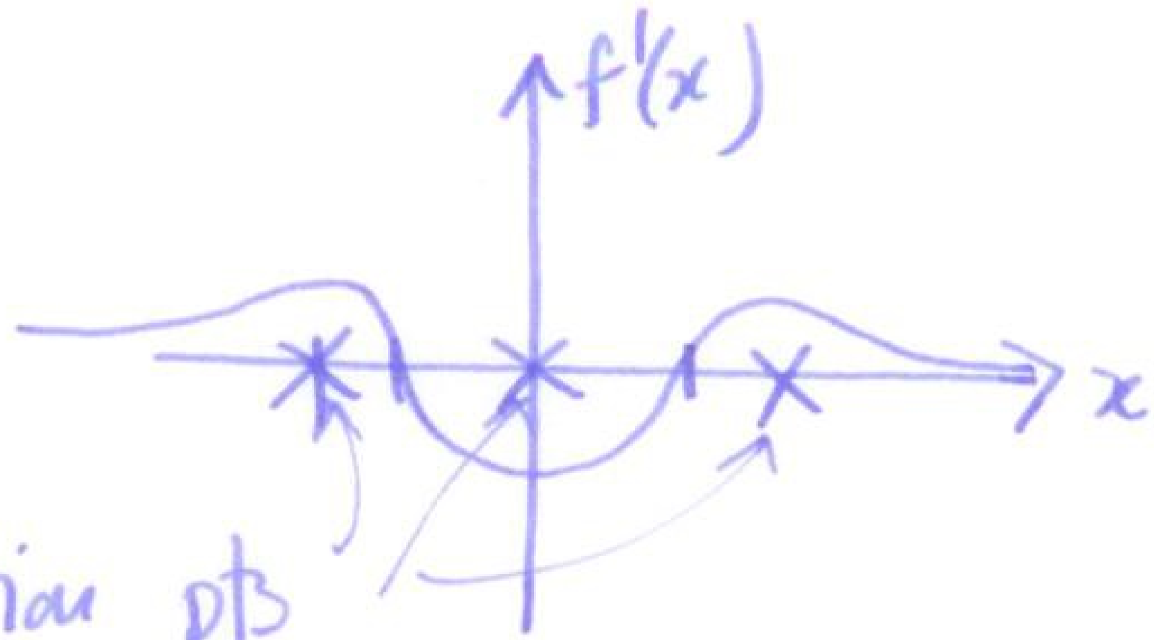
a) $[-\frac{1}{2}, \frac{1}{2}]$

b) $[-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty]$

c) 0

d) 0

e)



f) inflection pts

Q4

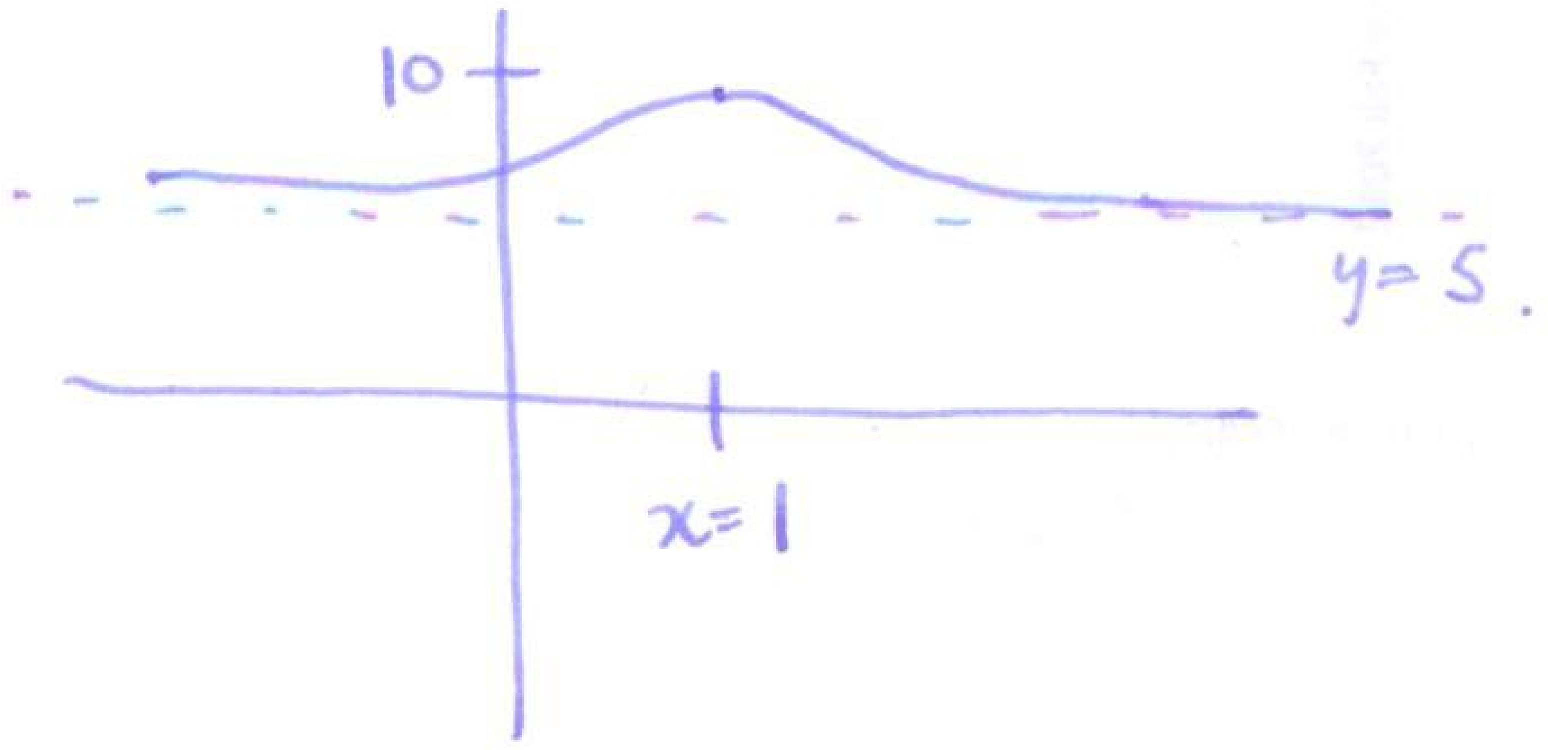
a) F

b) T

c) F

d) F

Q5



Q6

$f'(x) > 0$ for all x so function increasing, so minimum on $[1,4]$ at $x = 1$.

Q7 $g(x) = (x^2 - x)e^{-x}$

$g'(x) = -(x^2-x)e^{-x} + (2x-1)e^{-x} = (-x^2+3x-1)e^{-x}$

a) critical pts $g'(x) = 0$: $x^2 - 3x + 1 \Rightarrow$ ~~the~~ $x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$

$g''(x) = -(-x^2+3x-1)e^{-x} + (-2x+3)e^{-x}$

$= (x^2 - 5x + 2)e^{-x}$

$g''\left(\frac{3+\sqrt{5}}{2}\right) \approx -4.2 < 0$ so $\left(\frac{3+\sqrt{5}}{2}, 0.3\right)$ local max

$g''\left(\frac{3-\sqrt{5}}{2}\right) \approx 0.2 > 0$ so $\left(\frac{3-\sqrt{5}}{2}, -0.16\right)$ local min

b) $g'(x) \geq 0$ on $\left[\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right]$ increasing

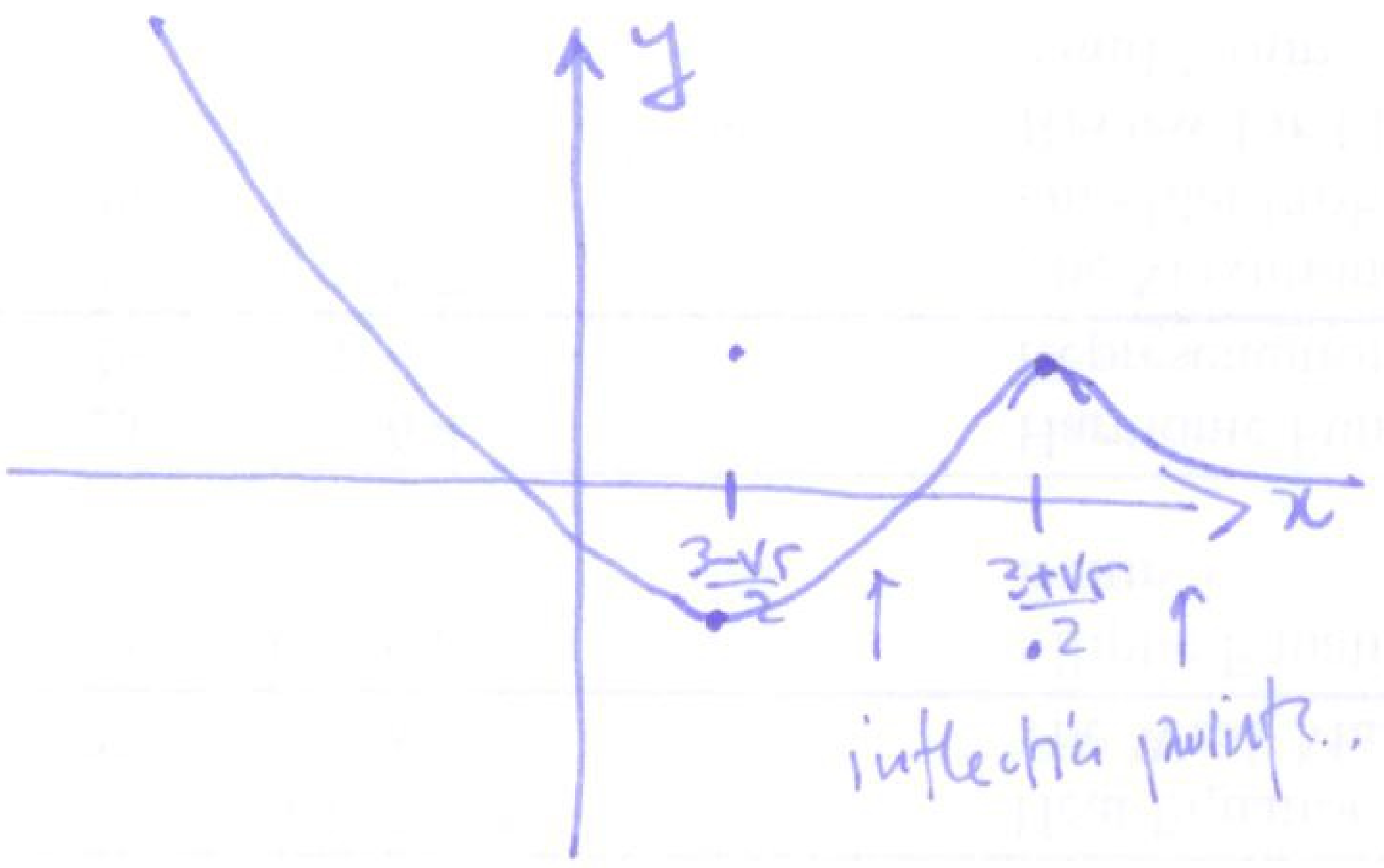
$g'(x) < 0$ on $\left(-\infty, \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}, \infty\right)$ decreasing

c) $g''(x) = 0$ $x = \frac{5 \pm \sqrt{25-8}}{2} = \frac{5 \pm \sqrt{17}}{2}$

d) $g''(x) > 0$ on $\left(-\infty, \frac{5-\sqrt{17}}{2}\right) \cup \left(\frac{5+\sqrt{17}}{2}, \infty\right)$

$g''(x) \leq 0$ on $\left[\frac{5-\sqrt{17}}{2}, \frac{5+\sqrt{17}}{2}\right]$

e)



Q8 a) $f(x) = \dfrac{x}{x^3+1}$ horizontal asymptotes $\lim\limits_{x\to\infty} \dfrac{x}{x^3+1} = 0$

$\lim\limits_{x\to-\infty} \dfrac{x}{x^3+1} = 0.$

vertical asymptotes: $x^3+1=0 \quad x=-1$, $x = \dfrac{1\pm\sqrt{1-4}}{2}$

$(x+1)(x^2-x+1)=0$ no other real roots.

b) $f'(x) = \dfrac{(x^3+1).1-(3x^2).x}{(x^3+1)^2}$ $\qquad f'(x)=0$

$x^3+1-3x^3=0.$

$2x^3-1=0$

$f'(x) = \dfrac{1-2x^3}{(x^3+1)^2}$ $\qquad x = \dfrac{-1}{\sqrt[3]{2}}$ no other real roots.

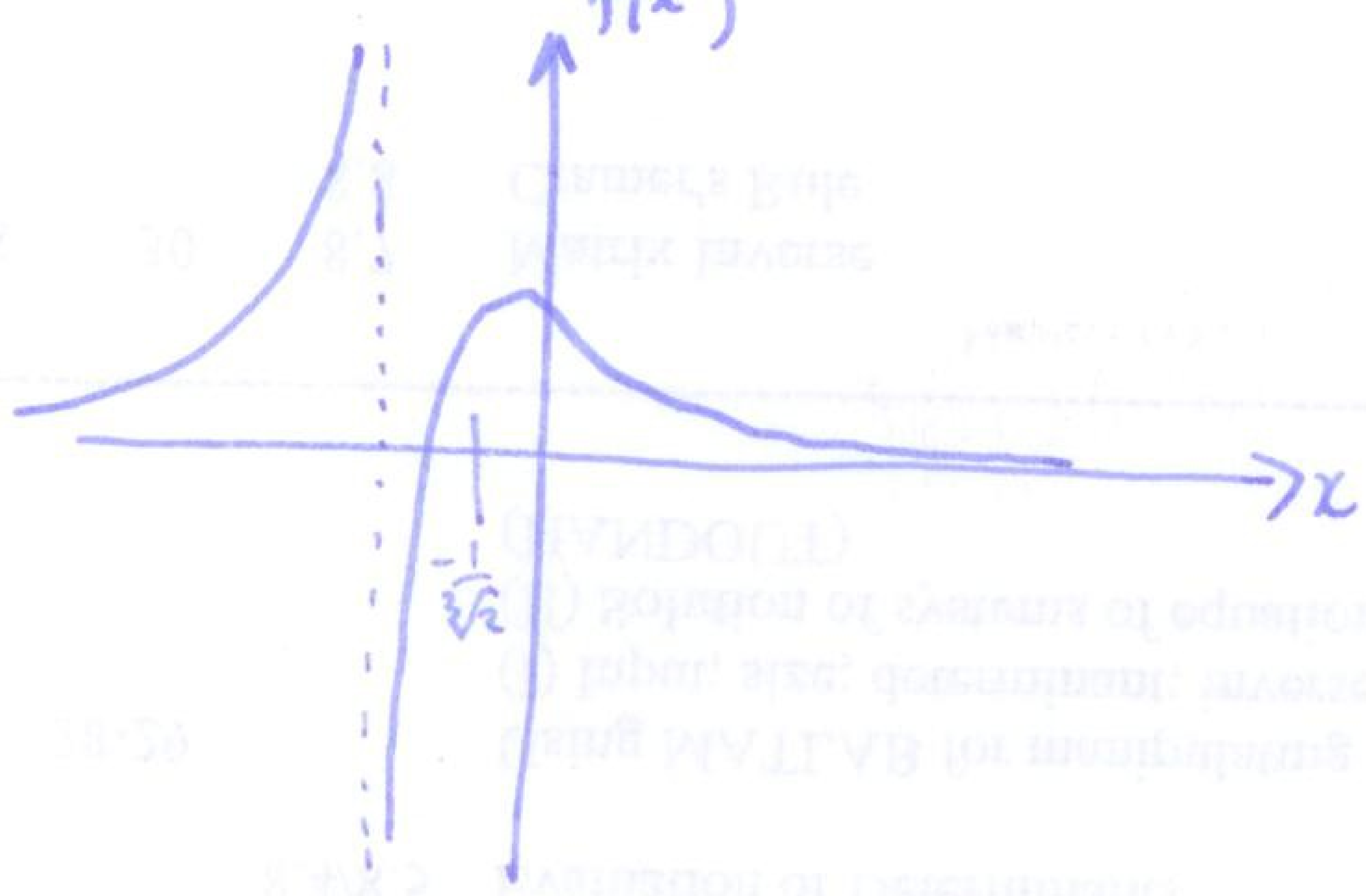
c) $f'(x) > 0$ on $(-\infty, -\frac{1}{\sqrt[3]{2}})$ increasing

$f'(x) < 0$ on $(-\frac{1}{\sqrt[3]{2}}, \infty)$ decreasing

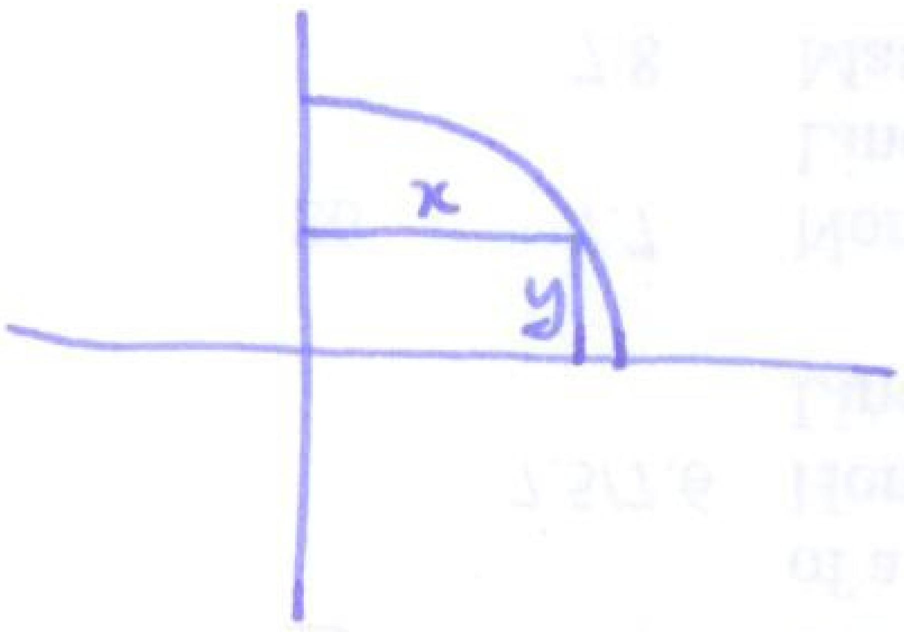
d) $f''(x) = \dfrac{(x^3+1)^2(-6x^2) - 2(x^3+1)3x^2(1-2x^3)}{(x^3+1)^4}$

$f''(-\frac{1}{\sqrt[3]{2}}) = \dfrac{(+)(-)<0}{+} < 0$

$f''(-\frac{1}{\sqrt[3]{2}}) = 0$ ~~no information~~ local max e)



Q9



$$A = xy \qquad x^2+y^2=1$$

$$\Rightarrow y = \sqrt{1-x^2}$$

$$A = x\left(1-x^2\right)^{1/2}$$

$$\frac{dA}{dx} = 1\left(1-x^2\right)^{1/2} + x\tfrac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)$$

$$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$$

solve $\frac{dA}{dx}=0 \Rightarrow x^2 = 1-x^2 \Rightarrow x^2 = \frac{1}{2} \quad x = \frac{1}{\sqrt{2}}$

largest area $\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$.

Q10

a) $$\lim_{x\to\infty} \frac{6x^5 - 12x^4 - 24}{2x^5 + 30} = \lim_{x\to\infty} \frac{6 - 12/x - 24/x^5}{2 + 30/x^5} = 3.$$

b) $$\lim_{x\to\infty} \frac{x^2}{1-x^2} = \lim_{x\to\infty} \frac{1}{-1 + 1/x^2} = -1$$

c) $$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+5}} \underset{\text{l'Hôpital}}{=} \frac{3}{\sqrt{1 + 5/x^2}} = 3.$$

d) $$\lim_{x\to 0} \frac{e^{2x^2} - 1}{1 - \cos(3x)} \underset{\text{l'Hôpital}}{=} \lim_{x\to 0} \frac{e^{2x^2} \cdot 4x}{\sin(3x) \cdot 3}$$

$$\underset{\text{l'Hôpital}}{=} \lim_{x\to 0} \frac{e^{2x^2} \cdot 4 + e^{2x^2} \cdot 4x \cdot 4x}{\cos(3x) \cdot 9} = \frac{4}{9}.$$