Linear Algebra Spring 10 Sample midterm 2

(1) (10 points)
(a) Circle one: A linearly independent set in $\mathbb{R}^n$ that is NOT a basis has how many vectors?
   AT LEAST $n$ / AT LEAST $n + 1$ / AT MOST $n - 1$ / AT MOST $n$
(b) Circle one: A spanning set in $\mathbb{R}^n$ that is NOT a basis has how many vectors?
   AT LEAST $n$ / AT LEAST $n + 1$ / AT MOST $n - 1$ / AT MOST $n$
(c) Circle one: For $A_{7 \times 4}$, if the columns of $A$ are linearly independent, then $Ax = 0$ has
   0 / 1 / $\infty$-many solutions.
(d) If the rows of $A_{3 \times 5}$ are linearly independent, what is $\text{nullity}(A)$?
(e) If $\text{Im}(A_{3 \times 5})$ is the plane $x - 4y + z = 0$, what are rank($A$) and nullity($A$)?

(2) (15 points)

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
4 & 7 & 1 \\
2 & 5 & -1
\end{bmatrix}
\]

(a) Find the reduced row-echelon form, $rref(A)$.
(b) Find the rank and nullity of $A$. Justify!
(c) Find a basis for the column space of $A$.

(3) (15 points)

\[
A = \begin{bmatrix}
1 & 3 & -2 & 1 & 1 \\
2 & 6 & -1 & -1 & 2 \\
2 & 6 & 1 & -3 & -1
\end{bmatrix}
\]

\[
rref(A) = \begin{bmatrix}
1 & 3 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(a) Find the rank and nullity of $A$.
(b) Find a basis for $\text{Im}(A)$.
(c) Find a basis for $\text{Ker}(A)$.

(4) (15 points)

(a) Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $L(x) = Ax$, where

\[
A = \begin{bmatrix}
2 & 1 \\
3 & -2 \\
-1 & 3
\end{bmatrix}
\]

Is the vector $(-1, -2, 2)$ in the range of $L$? Justify.
(b) Let \( L(x, y, z) = (5x + 3y - 2z, 4x - y + 3z) \). Find the standard matrix that represents \( L \).

(5) (25 points)

Let \( S = \{ u_1, u_2 \} \) and \( T = \{ u_1, u_3 \} \) be two bases for \( \mathbb{R}^2 \), where

\[
  u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}
\]

(a) Verify that \( S \) is a basis.

(b) Find the coordinate vector of \( v = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \) with respect to the basis \( S \).

(c) Find the coordinate vector of \( v = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \) with respect to the basis \( T \).

(d) Find the transition matrix \( P \) from \( S \) to \( T \).

(e) Verify that \([v]_S \) and \([v]_T \) are related by the transition matrix.

(6) (20 points)

(a) A student’s scores are given as a vector \( u \), and each score is worth a certain percentage of the total, given by a vector \( v \). For example, \( u = (82, 75, 89, 9, 6, 8, 8, 7, 9) \) and \( v = (20, 20, 30, 5, 5, 5, 5, 5, 5) \). Write a general formula for the total student score \( S(u, v) \) out of 100. (Use \( u \) and \( v \) in general, not just for this example!)

(b) Suppose \( u \) is orthogonal to both \( v \) and \( w \). Show that \( u \) is orthogonal to any vector \((rv + sw)\) for any scalars \( r \) and \( s \).

(c) Write down the matrix for a linear transformation of \( \mathbb{R}^2 \) that makes a \( \pi/2 \) rotation and doubles the length. (Hint: Draw a picture.)

(d) Write the matrix for a linear transformation of \( \mathbb{R}^3 \) that makes a \( \pi/2 \) rotation in the \( xz \)-plane.

(7) (20 points)

An \( n \times n \) magic square is an \( n \times n \) matrix of real numbers such that the sum of the entries in any row or column, and along either of the main diagonals yields the same answer. Express the set \( M \) of magic squares as the kernel of a linear map \( \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{2n+1} \), and deduce that it is a vector space. What is its dimension? Find a basis for the space of \( 3 \times 3 \) magic squares.