Linear Algebra Spring 10 Sample midterm 2

(1) (10 points)
  (a) Circle one: A linearly independent set in $\mathbb{R}^n$ that is NOT a basis has how many vectors?
  AT LEAST n / AT LEAST n + 1 / AT MOST n - 1 / AT MOST n
  (b) Circle one: A spanning set in $\mathbb{R}^n$ that is NOT a basis has how many vectors?
  AT LEAST n / AT LEAST n + 1 / AT MOST n - 1 / AT MOST n
  (c) Circle one: For $A_{7\times 4}$, if the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{0}$ has
  0 / 1 / ∞-many solutions.
  (d) If the rows of $A_{3\times 5}$ are linearly independent, what is nullity(A)?
  (e) If Im($A_{3\times 5}$) is the plane $x-4y+z=0$, what are rank(A) and nullity(A)?

(2) (15 points)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 7 & 1 \\ 2 & 5 & -1 \end{bmatrix}$$

  (a) Find the reduced row-echelon form, rref(A).
  (b) Find the rank and nullity of A. Justify!
  (c) Find a basis for the column space of A.

(3) (15 points)

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 & 1 \\ 2 & 6 & -1 & -1 & 2 \\ 2 & 6 & 1 & -3 & -1 \end{bmatrix} \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

  (a) Find the rank and nullity of A.
  (b) Find a basis for Im(A).
  (c) Find a basis for Ker(A).

(4) (15 points)
  (a) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \\ -1 & 3 \end{bmatrix}$$

  Is the vector $(-1, -2, 2)$ in the range of L? Justify.

(b) Let $L(x, y, z) = (5x + 3y - 2z, 4x - y + 3z)$. Find the standard matrix that represents L.

(5) (25 points)

Let $S = \{u_1, u_2\}$ and $T = \{u_1, u_3\}$ be two bases for $\mathbb{R}^2$, where

$$u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(a) Verify that S is a basis.

(b) Find the coordinate vector of $v = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ with respect to the basis S.

(c) Find the coordinate vector of $v = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ with respect to the basis T.

(d) Find the transition matrix P from S to T.

(e) Verify that $[v]_S$ and $[v]_T$ are related by the transition matrix.

(6) (20 points)

(a) A student's scores are given as a vector u, and each score is worth a certain per- centage of the total, given by a vector v. For example, $u = (82, 75, 89, 9, 6, 8, 8, 7, 9)$ and $v = (20, 20, 30, 5, 5, 5, 5, 5, 5)$. Write a general formula for the total student score $S(u, v)$ out of 100. (Use u and v in general, not just for this example!)

(b) Suppose u is orthogonal to both v and w. Show that u is orthogonal to any vector $(rv + sw)$ for any scalars r and s.

(c) Write down the matrix for a linear transformation of $\mathbb{R}^2$ that makes a $\pi/2$ rotation and doubles the length. (Hint: Draw a picture.)

(d) Write the matrix for a linear transformation of $\mathbb{R}^3$ that makes a $\pi/2$ rotation in the xz-plane.

(7) (20 points)

An $n \times n$ magic square is an $n \times n$ matrix of real numbers such that the sum of the entries in any row or column, and along either of the main diagonals yields the same answer. Express the set M of magic squares as the kernel of a linear map $\mathbb{R}^{n^2} \to \mathbb{R}^{2n+1}$, and deduce that it is a vector space. What is its dimension? Find a basis for the space of 3×3 magic squares.