Linear Algebra Sample midterm 2 Solutions

Q1 a) at most $n-1$

b) at least $n+1$

c) 1 solution

d) 2

e) 3

Q2 a) $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 7 & 1 \\ 2 & 5 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

b) $\text{rank}(A) = 2$, ~~then~~ row space A = row space $\text{rref}(A)$, is 2dimensional.

$\text{nullity}(A) = 1$, one free variable, i.e. 1 col w/out leading 1.

c) basis for column space $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 5 \end{bmatrix}$ (cols with leading 1's).

Q3 a) $\text{rank}(A) = 3$ $\text{nullity}(A) = 2$

b) $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

c) $x_5 = 0$

$x_4 = s$

$x_3 = x_4 = s$

$x_2 = t$

$x_1 = s - 3t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} s-3t \\ t \\ s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

basis for $\ker(A)$: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Q4

a) solve $A\underline{x} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ ⊛ $\quad \left[\begin{array}{cc|c} 2 & 1 & -1 \\ 3 & -2 & -2 \\ -1 & 3 & 2 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 7 & 3 \\ 0 & 7 & 4 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 7 & 3 \\ 0 & 0 & 1 \end{array}\right]$ no. equation ⊛ has no solutions.

b) L has standard matrix $\begin{bmatrix} 5 & 3 & -2 \\ 4 & -1 & 3 \end{bmatrix}$

Q5 a) $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ row equivalent to I_2, so a basis.

b) solve $c_1\underline{u_1} + c_2\underline{u_2} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ -1 & 2 & 4 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 3 & 3 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right] \quad \begin{matrix} c_1 = -2 \\ c_2 = 1 \end{matrix}$ i.e. $\left[\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right]_S = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

c) solve $c_1\underline{u_1} + c_3\underline{u_3} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ -1 & 2 & 4 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \quad \begin{matrix} c_1 = 2 \\ c_2 = 3. \end{matrix}$

so $\left[\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right]_T = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

d) $P_{E\leftarrow S} = \begin{bmatrix}1&1\\-1&2\end{bmatrix}$ (E standard basis)

$P_{E\leftarrow T} = \begin{bmatrix}1&-1\\-1&2\end{bmatrix}$ $\quad P_{T\leftarrow E} = P^{-1}_{\cancel{T}\leftarrow E} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$

$P_{T\leftarrow S} = \cancel{P_{E\leftarrow S}}\; P_{T\leftarrow E} P_{E\leftarrow S} \quad \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\-1&2\end{bmatrix} = \begin{bmatrix}1&4\\0&3\end{bmatrix}$

e) $\begin{bmatrix}1&4\\0&3\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$ ✓

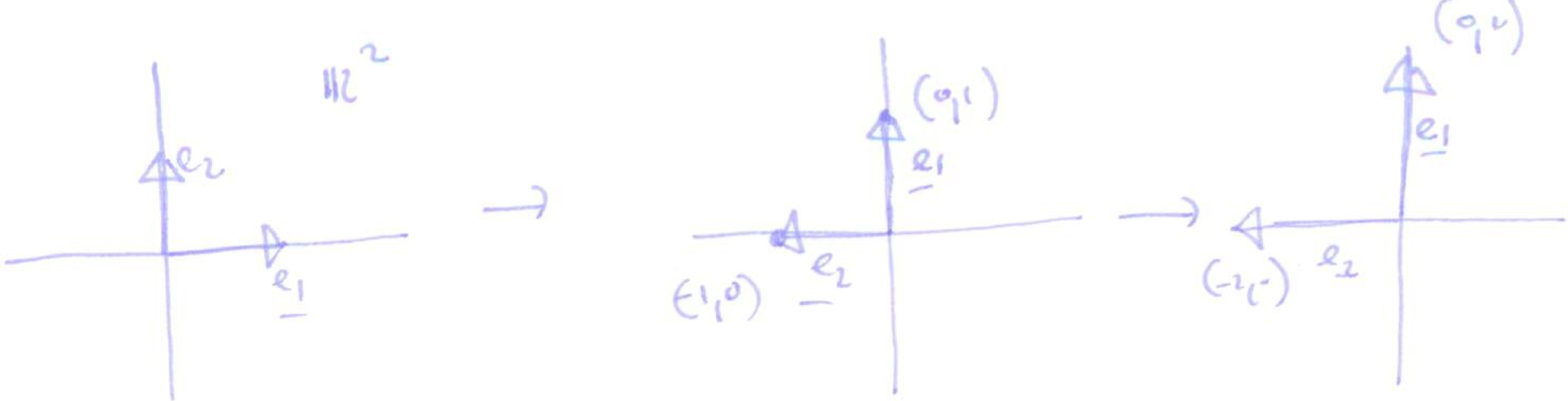
Q6 a) $s(\underline{u},\underline{v}) = \underline{u}.\underline{v}.\frac{1}{100}$.

b) $\underline{u}.\underline{v} = 0$, $\underline{u}.\underline{w} = 0$

$\underline{u}.(r\underline{v} + s\underline{w}) = r\underline{u}.\underline{v} + s\underline{u}.\underline{w} = r.0 + s.0 = 0.$

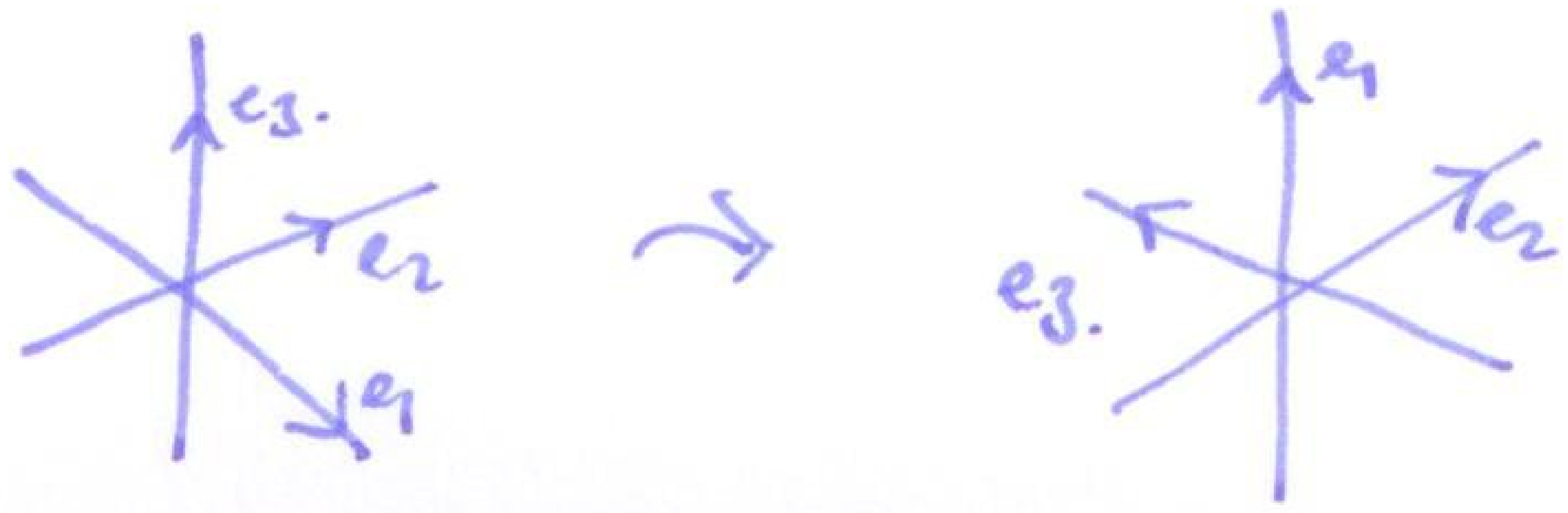
$\therefore$ $\underline{u}$ orthogonal to $r\underline{v}+s\underline{w}$.

c)



$\begin{bmatrix}0&-2\\2&0\end{bmatrix}$.

d)



$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$

Q7 $M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ then set row sums

$$r_i = a_{i1} + a_{i2} + \cdots + a_{in}$$

col sums $c_j = a_{1j} + a_{2j} + \cdots + a_{nj}$

diagonal $t = a_{11} + a_{22} + \cdots + a_{nn}$

cross ~~off~~diagonal $t^* = a_{1n} + a_{2n-1} + \cdots + a_{n1}$

then define $f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{2n+1}$ by

$$[a_{ij}] = M \mapsto (r_1 - t^*, r_2 - t^*, \ldots r_n - t^*, c_1 - t^*, \ldots, c_n - t^*, t - t^*)$$

so magic squares $= \ker(f)$.

note that $r_1 - t^* + r_2 - t^* + \cdots + r_n - t^* = c_1 - t^* + \cdots + c_n - t^*$

so image(f) has dimension $\leq 2n$

$\Rightarrow$ $\ker(f)$ has dimension $\geq n^2 - 2n$

(sketch)

Let $N = \{$ squares with row/col condition but no diagonal condition $\}$.

$N_0 = \{$ squares with row/col sums $= 0 \}$ so $N_0 \subset N$

$$\begin{tikzcd}
 & M = \ker(t, t^*) \arrow[d] & \\
N_0 \arrow[r, hook] \arrow[d, "\beta"] & N \arrow[r, "\text{row/col sum}"] \arrow[d, "t, t^*"] & \mathbb{R} \\
M_{(n-1)(n-1)} & \mathbb{R}^2 &
\end{tikzcd}$$

β is map which takes top right $(n-1) \times (n-1)$ matrix

show onto, injective

then $\dim N_0 = (n-1)^2$

$\dim N = (n-1)^2 + 1$

now show map $(t, t^*) : N \to \mathbb{R}^2$ onto

so $\dim M = (n-1)^2 + 1 - 2 = n^2 - 2n$.