Linear Algebra Spring 10 Midterm 1

Name: Solutions

1	15	
2	21	
3	15	
4	15	
5	15	
6	14	
7	10	
	105	

(1) (15 points) Consider the following linear system:

$$\begin{cases} x_1 - x_2 + x_4 = 2 \\ x_1 - x_3 + 2x_4 = 0 \\ -x_2 + x_3 + x_4 = -6 \end{cases}$$

(a) Write its associated augmented matrix.

(b) Reduce the matrix to its reduced row-echelon form.

(c) Use this procedure to solve the system.

a) $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 & -6 \end{array}\right]$

b) $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 & -2 \\ 0 & -1 & 1 & 1 & -6 \end{array}\right]$ $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 2 & -8 \end{array}\right]$

$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$ $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 6 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$

$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 8 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$

c) $x_4 = -4$

$x_3 = t$

$x_2 - x_3 = 2 \qquad x_2 = 2+t$

$x_1 - x_3 = 8 \qquad x_1 = 8+t$

$$\underset{\sim}{x} = \begin{bmatrix} 8+t \\ 2+t \\ t \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(2) (21 points) State whether each of the following statements is true or false.

(a) If A and B are $n \times n$ matrices and A is singular, then (AB) is singular.

(b) If $A^2 = I_n$, then $A = I_n$ or $A = -I_n$.

(c) If A and B are diagonal matrices then $AB = BA$.

(d) A homogeneous system with more variables than equations has a finite number of solutions.

(e) Let 0 be the zero matrix. If A is row equivalent to 0, then $A = 0$.

(f) If $A^2 = 0$ then $A = 0$.

(g) A 3×4 matrix in reduced row echelon form can contain a row consisting of $\begin{bmatrix} 0 & 0 & 0 & 2 \end{bmatrix}$

a) True

b) False

c) True.

d) False

e) True

f) False

g) False

(3) (15 points)

(a) Solve the following linear system:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

(b) Give an example of a 3×4 matrix in reduced row-echelon form that has one row $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ and has two entries consisting of the number 4.

(c) If A is an invertible matrix such that $A^2 = A$, compute the determinant $|A|$. Show your work!

a) $w = t$

$z + 2w = 4 \qquad z = 4 - 2t$

$y = 3$

$x - w = -1 \qquad x = -1 + t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1+t \\ 3 \\ 4-2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c) $\det(A^2) = \det(A)\det(A)$

so $A^2 = A \Rightarrow \det(A)^2 = \det(A)$

$\det(A)^2 - \det(A) = 0$

$\det(A)(\det(A) - 1) = 0 \qquad$ so $\det(A) = 0, 1$

A invertible $\Rightarrow \det(A) \neq 0 \qquad$ so $\det(A) = 1$.

(4) (15 points) Evaluate the following determinants:

(a)

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

(b) If A, B are 3×3 matrices with $|A| = 2$ and $|B| = 3$, compute $|2AB|$.

(c) If A, B are 3×3 matrices with $|A| = 2$ and $|B| = 3$, compute $|A^4 B^T A^{-1}|$.

a) $\det = 1.2.3.4.5.6 = 720$

b) $\det(AB) = \det(A)\det(B)$ $\det(\lambda A) = \lambda^n \det(A)$ if A $n \times n$.

so $\det(2AB) = 2^3 . 2 . 3 = 48$

c) $\det(A^4 B^T A^{-1}) = 2^4 . 3 . \frac{1}{2} = 24$.

(5) (15 points) Justify the following statements with a short general argument.

(a) If $\det(A) \neq 0$ then $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(b) If $A^{-1} = A^T$ then $\det(A^{-1}) = \pm 1$.

(c) If A is any $n \times n$ matrix then $(A + A^T)$ is symmetric.

a) $\det(A) \neq 0 \Rightarrow A^{-1}$ exists.

so $A^{-1}A\underline{x} = A^{-1}\underline{b}$ $\quad \underline{x} = A^{-1}\underline{b}$, unique solution.

b) $\det(A^T) = \det(A)$ so $\dfrac{1}{\det(A)} = \det(A) \Rightarrow \det(A)^2 = 1$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$ $\quad \Rightarrow \det(A) = \pm 1$.

c) $\left.\begin{aligned} [(A+A^T)_{ij}] &= a_{ij} + a_{ji} \\ [(A+A^T)_{ji}] &= a_{ji} + a_{ij} \end{aligned}\right\}$ same so $A+A^T$ symmetric.

(6) (14 points)

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & a \\ 0 & -1 & 2 \end{pmatrix}$$

(a) For which values of a is A invertible?

(b) Use elementary operations to find the inverse of A when $a = -1$.

a)

$$\det(A) = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - a \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 2 + a$$

invertible as long as $a \neq -2$.

b)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

(7) (10 points)

Suppose a linear system corresponds to two planes in $\mathbb{R}^3$ which intersect in a line. Write down a matrix in reduced row echelon form which could correspond to this situation.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$