Linear Algebra Spring 10 Final

Name: Solutions

- You may use a calculator, but no notes.

1	10	
2	10	
3	15	
4	10	
5	10	
6	20	
7	20	
8	10	
9	10	
	115	

(1) (10 points) Find all solutions to the following system of linear equations.

$$\begin{aligned} x_1 + x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + x_2 + 3x_3 + x_4 &= 0 \\ x_1 + x_3 + x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ & & t & \end{array}$$

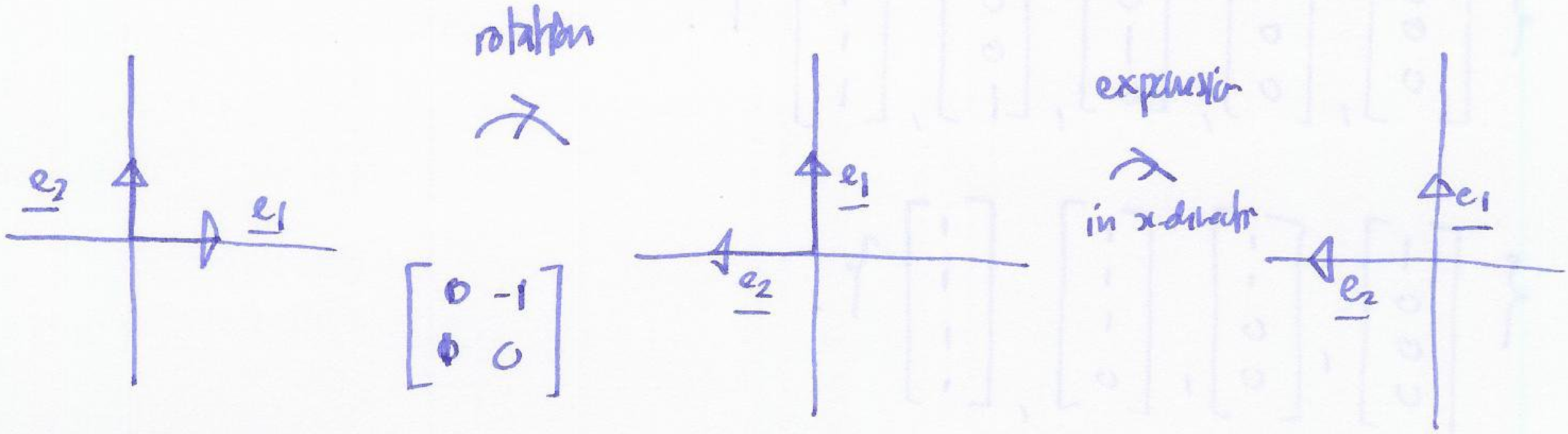
$x_4 = 0$

$x_2 + t = 0$

$x_1 + t = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

(2) (10 points) Write down a matrix for a linear transformation of $\mathbb{R}^2$ which rotates by $\pi/2$ anticlockwise about the origin, and then doubles lengths in the x-direction.



$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

answer : $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

(3) (15 points)

(a) Write down a spanning set for $\mathbb{R}^4$ which is not a basis.

(b) Write down a basis for $\mathbb{R}^4$ which is not orthogonal.

(c) Write down a set of three vectors which span a two-dimensional subspace of $\mathbb{R}^4$.

a) $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}.$

b) $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}.$

c) $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \right\}.$

(4) (10 points)

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Start using the Gram-Schmidt process to find an orthogonal basis, but only find the first two vectors, don't bother to find the third.

$$\underline{w_1} = \underline{v_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\underline{w_2} = \underline{v_2} - \lambda \underline{w_1} \quad \text{where} \quad \lambda = \frac{\underline{w_1} \cdot \underline{v_2}}{\underline{w_1} \cdot \underline{w_1}} = \frac{-2}{3}$$

$$\underline{w_2} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ 4/3 \end{bmatrix}$$

(5) (10 points)

Let $V = \text{Span}\{\underset{\underline{v_1}}{(-1,2,1)}, \underset{\underline{v_2}}{(1,3,-1)}, \underset{\underline{v_3}}{(-3,1,3)}\}$ in $\mathbb{R}^3$.

(a) What is the dimension of V?

(b) Find a basis for $V^{\perp}$.

$$\begin{bmatrix} -1 & 2 & 1 \\ 1 & 3 & -1 \\ -3 & 1 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) row span doesn't change under row operations so $\dim V = 2$.

b) solve:

$$\left.\begin{aligned} \underline{v_1} \cdot \underline{x} &= 0 \\ \underline{v_2} \cdot \underline{x} &= 0 \\ \underline{v_3} \cdot \underline{x} &= 0 \end{aligned}\right\} \text{ same linear system as above.}$$

$x_3 = t$

$x_2 = 0$

$x_1 = t$

so sol$^{\text{ns}}$ $t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ basis for $V^{\perp}$ is $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(6) (20 points)

$$A = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors for A.
(c) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.

a) $\begin{vmatrix} 7-\lambda & 5 \\ -10 & -8-\lambda \end{vmatrix} = -(7-\lambda)(8+\lambda) + 50 = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3)$

eigenvalues $\lambda_1 = 2$, $\lambda_2 = -3$

b) $\begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix}$ $\underline{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix}$ $\underline{v_2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

c) $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ $P^{-1} = -\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

check: $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

(7) (20 points)
Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x, y) = (x - 3y, 2x + y)$. Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^2$, and let T be the standard basis for $\mathbb{R}^2$.
(a) Find the matrix for L with respect to T.
(b) Find the matrix for the change of basis from S to T.
(c) Find the matrix for L with respect to S. Don't worry if its not diagonal.

a) $\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} = A$

b) $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} = P$

c)
$$\begin{tikzcd}
\mathbb{R}^2_T \arrow[r, "L", "A"'] & \mathbb{R}^2_T \\
\mathbb{R}^2_S \arrow[u, "P"] \arrow[r, "B"'] & \mathbb{R}^2_S \arrow[u, "P"']
\end{tikzcd}$$

$B = P^{-1}AP$

$$\begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} +1 & 4 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}.$$

$$= \begin{bmatrix} 14 & -5 \\ 35 & -12 \end{bmatrix}$$

(8) (10 points) If A is a non-singular $n \times n$ matrix such that $A^{-1} = A^T$, what can you say about the determinant of A?

$\det(AB) = \det(A)\det(B)$

$\det(A^T) = \det(A)$

$AA^T = I$

so $\det(A)^2 = \det(I) = 1$

so $\det(A) = \pm 1$

(9) (10 points) Let A be a 3×5 matrix such that the sum of the rows add up to the zero vector.

(a) What can you say about the column rank of A?

(b) If A determines a linear map $L : \mathbb{R}^5 \to \mathbb{R}^3$ given by $L(\mathbf{x}) = A\mathbf{x}$, what can you say about the kernel of L?

a) row rank ≤ 2 $\Rightarrow$ col rank ≤ 2.

b) $\underbrace{\text{rank}(A)}_{\leq 2} + \text{nullity}(A) = \#\text{cols} = 5.$

$\Rightarrow$ $\text{nullity}(A) \geq 3$