Linear Algebra Spring 10 Sample Final

(1) State whether the following statements are always true, sometime true, or never true. Give a brief indication of your reasoning.
  (a) An invertible matrix can be written as a product of symmetric matrices.
  (b) A homogeneous 3×5 linear system has a nontrivial solution.
  (c) If $\det(A) = 0$, then $\det(A + B) = \det(B)$.
  (d) If $\det(A) = 0$, then $\det(BA) = 0$.
  (e) A square matrix which has two identical columns is invertible.

(2) Justify three out of the following four statements with a short general argument:
  (a) If A is a non-singular $n \times n$ matrix then:
$$\det(A^{-1}) = \frac{1}{\det(A)}$$
  (b) If A and B are non-singular $n \times n$ matrices, then AB is also non-singular.
  (c) A non-singular matrix has a unique inverse.
  (d) If A and B are symmetric matrices, then AB is also symmetric.

(3) Write "impossible" or give an example of:
  (a) A 3×3 matrix with no zeros but which is not invertible.
  (b) A system with two equations and three unknowns that is inconsistent.
  (c) A system with two equations and three unknowns that has a unique solution.
  (d) A system with two equations and three unknowns that has infinitely many solutions.

(4) Consider the following system of equations:
$$\begin{cases} 2x_1 - x_2 + x_4 & = 0 \\ -x_1 + 2x_2 - x_3 & = 1 \\ -x_2 + 2x_3 & = 0 \end{cases}$$
Write its associated augmented matrix. Reduce the matrix to its row-echelon form. Use the procedure to solve the system.

(5) (a) Let:
$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 5 \\ -4 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 3 \\ 5 & 1 & 0 \end{bmatrix}$$
Compute $A + B$, AB, B^T, $\det(A)$, $\det(A^T)$ and $\det(3A)$.

(b) Use elementary operations to find the inverse of:

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(6) Consider the following two systems of equations:

$$\begin{cases} a_{11}x + 2y &= 1 \\ 1.1x + 2y &= 1 \end{cases}$$

$$\begin{cases} a_{11}x + 2y &= 1 \\ 2x + 1.1y &= 1 \end{cases}$$

where $a_{11} = 4$. User Cramer's rule to estimate the error in the solution arising from a 10% error in the measurement of a_{11}.

(7) (a) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $L(x) = Ax$, where

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -5 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

Is the vector $(1, 2, 3)$ in the range of L?

(b) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L(x, y) = (2x+3y, -2x+3y, x+y)$. Find the standard matrix representing L.

(8) Give a brief indication of your reasoning in the following questions.

(a) What can you say about the number of vectors for a spanning set of $\mathbb{R}^7$.

(b) If A is an $m \times n$ matrix with $\operatorname{rank}(A) < n$, then what can you say about the number of solutions of $Ax = 0$?

(c) If the columns of an $n \times n$ matrix A form an orthogonal set, then what can you say about the rank of A?

(d) If V has basis S, and T is obtained from S by the Gram-Schmidt process, what are the properties of T that are possibly different from S?

(e) Let A be a 4×3 matrix such that the sum of its columns equals 0. What is the largest possible value for the row rank of A?

(9)

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Find a subset of S that is a basis for the span of S. What is the dimension of the span of S?

(10) Let P_3 be the vector space of quadratic polynomials, with inner product given by

$$\int_{-1}^{1} p(t)q(t)dt.$$

Apply the Gram-Schmidt process to the basis $\{1, t, t^2\}$ to find an othonormal basis.

(11) (a) Show that trace is a linear map from the set of 2×2 matrices to $\mathbb{R}$.
(b) Consider the following basis for the set of symmetric 2×2 matrices.

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Write down a matrix representing trace with respect to this basis.
(c) Consider the alternate basis

$$T = \left\{ \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Write down a matrix representing the change of basis map from T to S, and use this to find a matrix for trace with respect to this new basis.

(12) Let S and T be bases for $\mathbb{R}^2$, where

$$S = \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$$

(a) Find the coordinate vector for $v = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$ with respect to S.
(b) Find T, given that $P_{S \leftarrow T} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

(13) Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvectors and eigenvalues for A.
(b) What are the eigenvalues for A^6? What does this tell you about A^6?

(14) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by $L(x, y) = (3x - y, -x + 2y)$.
(a) Write down a matrix for L with respect to the standard basis.

(b) Write down a matrix for L with respect to the basis

$$\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$