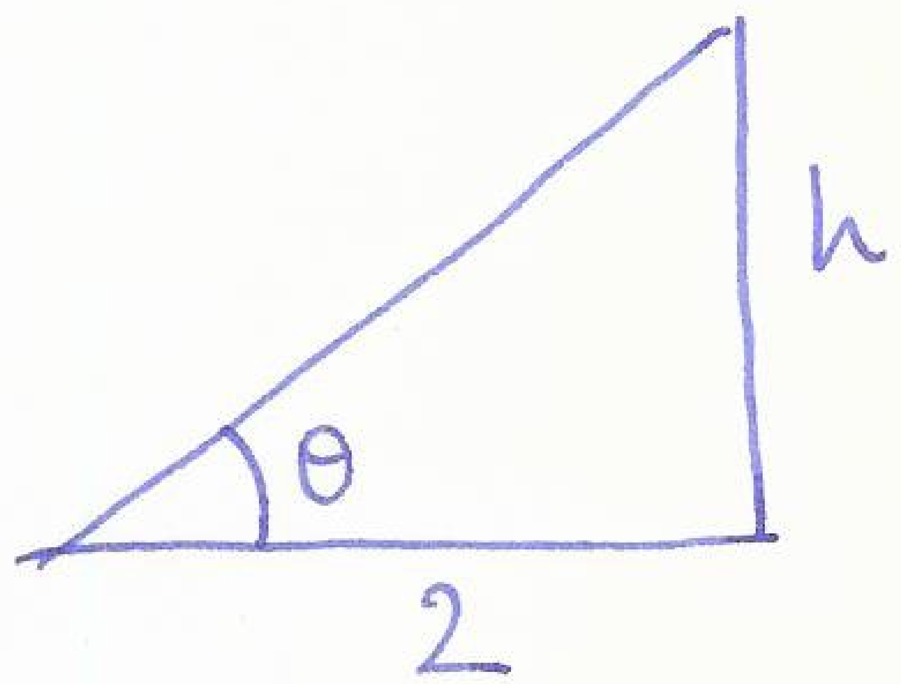
(1) (15 points) A hot air balloon rises vertically upwards from a distance of two miles away. When you see the balloon at an angle of $\pi/4$ radians, the angle is changing at a rate of 0.2 radians/s. How fast is the balloon rising?

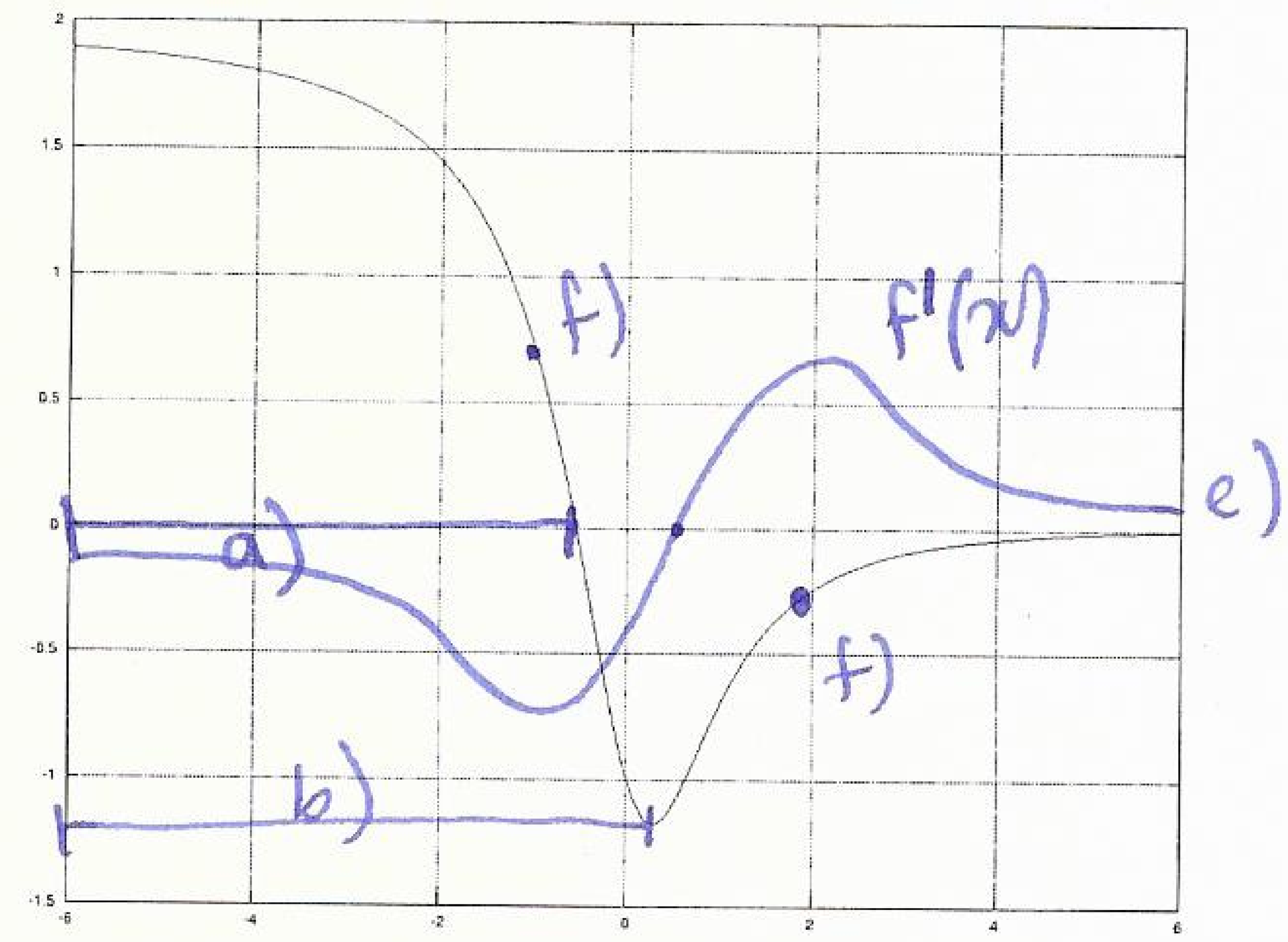


$$\tan\theta = \frac{h}{2}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$\frac{dh}{dt} = 2\sec^2\left(\frac{\pi}{4}\right) \cdot 0.2 = 0.8 \text{ m/s.}$$

(2) (25 points) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f(x) > 0$.
(b) Label all regions where $f'(x) < 0$.
(c) What is $\lim_{x\to\infty} f(x)$?
(d) What is $\lim_{x\to-\infty} f'(x)$?
(e) Sketch a graph of $f'(x)$ on the figure.
(f) Label the approximate locations of all points of inflection.

c) 0

d) 0

(3) (15 points) The value of $\tan x$ at $\pi/4$ is 1. Use a linear approximation to estimate $\tan(0.8)$. Do you consider this to be a good approximation?

$\Delta f \simeq f'\left(\frac{\pi}{4}\right)(0.8 - \pi/4)$ $\quad$ $\Delta x = 0.8 - \pi/4 \approx 0.0146.$

$f(x) = \tan(x)$

$f'(x) = \sec^2(x)$ $\quad$ $f'\left(\frac{\pi}{4}\right) = 2$ $\quad$ so $\Delta f \approx \overset{f'\left(\frac{\pi}{4}\right)\Delta x}{0.0292}$

so $\tan(0.8) \simeq 1 + 0.0292 = 1.0292$

actual value : $\tan(0.8) = 1.02964\ldots$

good approximation : error is $|1.0292 - 1.02964| \simeq 0.0004$

percentage error is $\dfrac{100 \times 0.0004}{1.02964\ldots} \simeq 0.04\%.$

(4) (25 points) Consider the function

$$f(x) = \frac{1}{x^2 - x \pm 2}$$

(a) Find all vertical and horizontal asymptotes of the function.
(b) Find all critical points of the function.
(c) Determine the intervals where $f(x)$ is increasing and decreasing.
(d) Use the 2nd derivative test to attempt to identify all local maxima and minima.
(e) Sketch the function and label all relative maxima and minima.

a) vertical asymptotes: $x^2 - x - 2 = (x-2)(x+1)$ vertical asymptotes at $x = 2, -1$

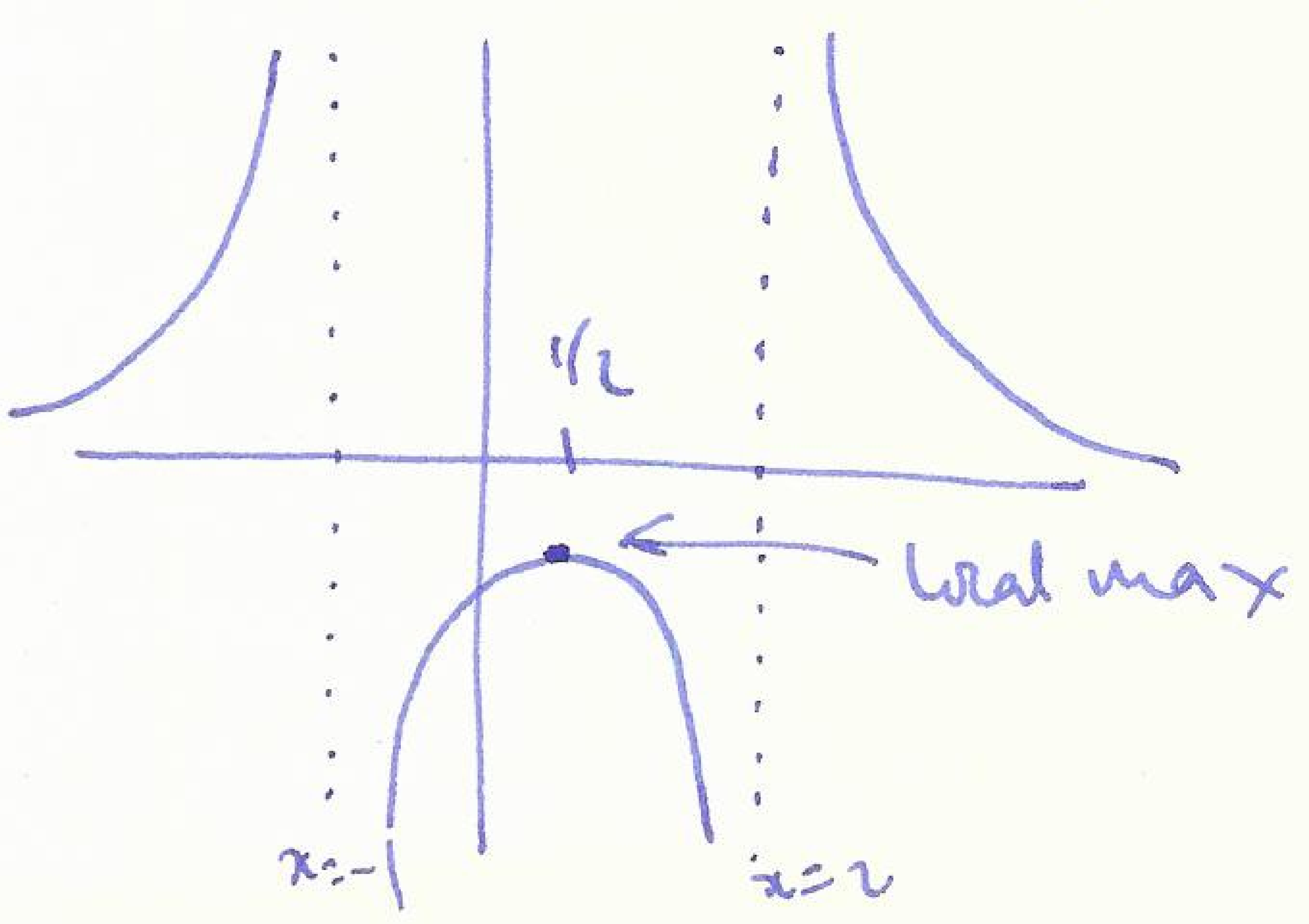
horizontal asymptotes: $\lim_{x\to\infty} f(x) = 0$ $\lim_{x\to-\infty} f(x) = 0$

b) $f'(x) = \dfrac{-(2x-1)}{(x^2-x-2)^2}$ critical point $x = 1/2$.

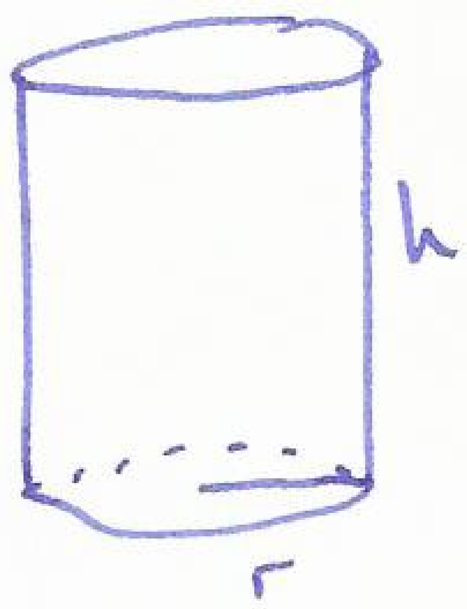
c) $f'(x) > 0$ increasing when $x < 1/2$
$f'(x) < 0$ decreasing when $x > 1/2$.

d) $f''(x) = \dfrac{-(x^2-x-2)^2(2) + 2(x^2-x-2)(2x-1)(2x-1)}{(x^2-x-2)^4}$

$f''(1/2) = -\dfrac{(+)(+) + (0)}{(+)} < 0$ so local max.



(5) (15 points) A cylindrical can of volume 1ft^3 is to be constructed, where the material for the top and bottom costs four times as much as the material for the sides. Find the dimensions which minimize the cost of the can.



volume $\quad V = 1 = \pi r^2 h \quad \Rightarrow h = \dfrac{1}{\pi r^2}.$

cost $\quad C = 4(2\pi r^2) + 2\pi r h$

$$C = 8\pi r^2 + \frac{2\pi r}{\pi r^2} = 8\pi r^2 + \frac{2}{r}$$

$$\frac{dC}{dr} = 16\pi r - \frac{2}{r^2} = 0 \Rightarrow r^3 = \frac{1}{8\pi} \quad r = \sqrt[3]{1/8\pi}$$

$r \simeq 0.34$

$h \simeq 2.75$

(6) (25 points) Compute the following limits. Show all work.

(a) $\lim_{x\to-\infty} \frac{2x^4-x^3+3}{(1-2x^2)^2}$

(b) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2-3x}}$

(c) $\lim_{x\to 0} \frac{e^{2x}-2}{\sin x}$

(d) $\lim_{x\to 0} \frac{1}{1-\cos x} - \frac{1}{x^2}$

(e) $\lim_{x\to 0} x^{\sin x}$

a) $= \lim_{x\to-\infty} \frac{2 - 1/x + 3/x^4}{(1/x^2 - 2)^2} = \frac{1}{2}$

b) $= \lim_{x\to\infty} \frac{1}{\sqrt{1-3/x}} = 1$

c) (l'Hôpital) $= \lim_{x\to 0} \frac{2e^{2x}}{\cos x} = 2$

d) $= \lim_{x\to 0} \frac{x^2 - 1 + \cos x}{x^2(1-\cos x)}$ (l'Hôpital) $= \lim_{x\to 0} \frac{2x - \sin x}{2x - 2x\cos x + x^2 \sin x}$

(l'Hôpital) $= \lim_{x\to 0} \frac{2 - \cos x}{2 - 2\cos x + 2x\sin x + 2x\sin x + x^2\cos x}$ $\frac{1}{0} \therefore \to \pm\infty$ as $x \to 0^{\pm}$.

e) $= \lim_{x\to 0} e^{\ln(x)\sin x}$ $\lim_{x\to 0} \ln(x)\sin(x) = \lim_{x\to 0} \frac{\ln(x)}{1/\sin(x)}$

(l'Hôpital) $= \lim_{x\to 0} \frac{1/x}{\frac{-\cos x}{\sin^2(x)}} = \lim_{x\to 0} -\frac{\sin x}{x}\tan x = 0$

So $\lim_{x\to 0} e^{\ln(x)\sin(x)} = 1$