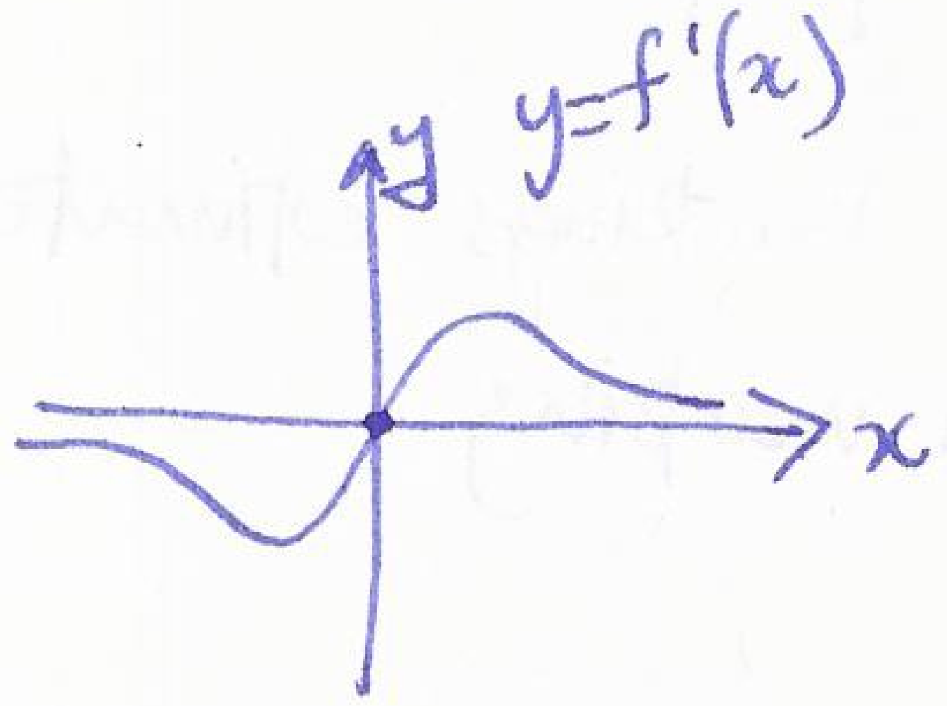
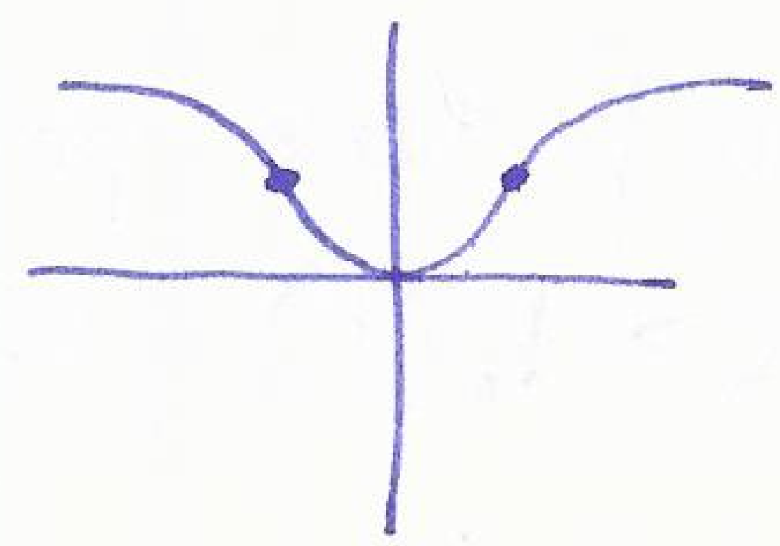
Sample midterm 3 Solutions

Q1 a) $(-6,0)$ b) $(0,6)$ c) 0 d) 0

e)



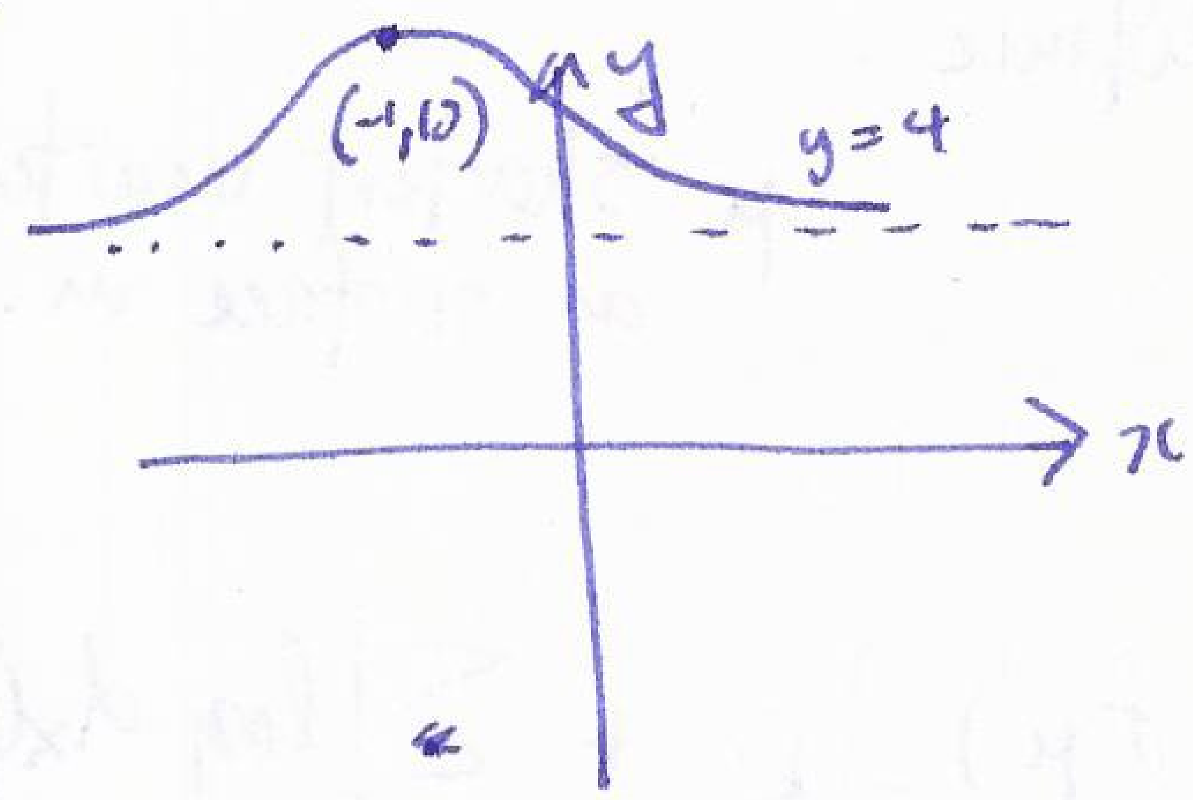
f)



Q2 a) false b) false as stated, interval should be $(0,5)$ then true.

c) false d) false.

Q3



Q4 a) $g'(x) = x^3 - 3x^2 = x^2(x-3)$ critical points $x=0,3$

$g''(x) = 3x^2 - 6x$

$g''(3) > 0$ so $(3, -27)$ is a local min

$g''(0) = 0$ no information, use first derivative test: $g'(0+\epsilon) < 0$, $g'(0-\epsilon) < 0$

not local max or min.

b) $g'(x)$ can only change sign at $0,3$.

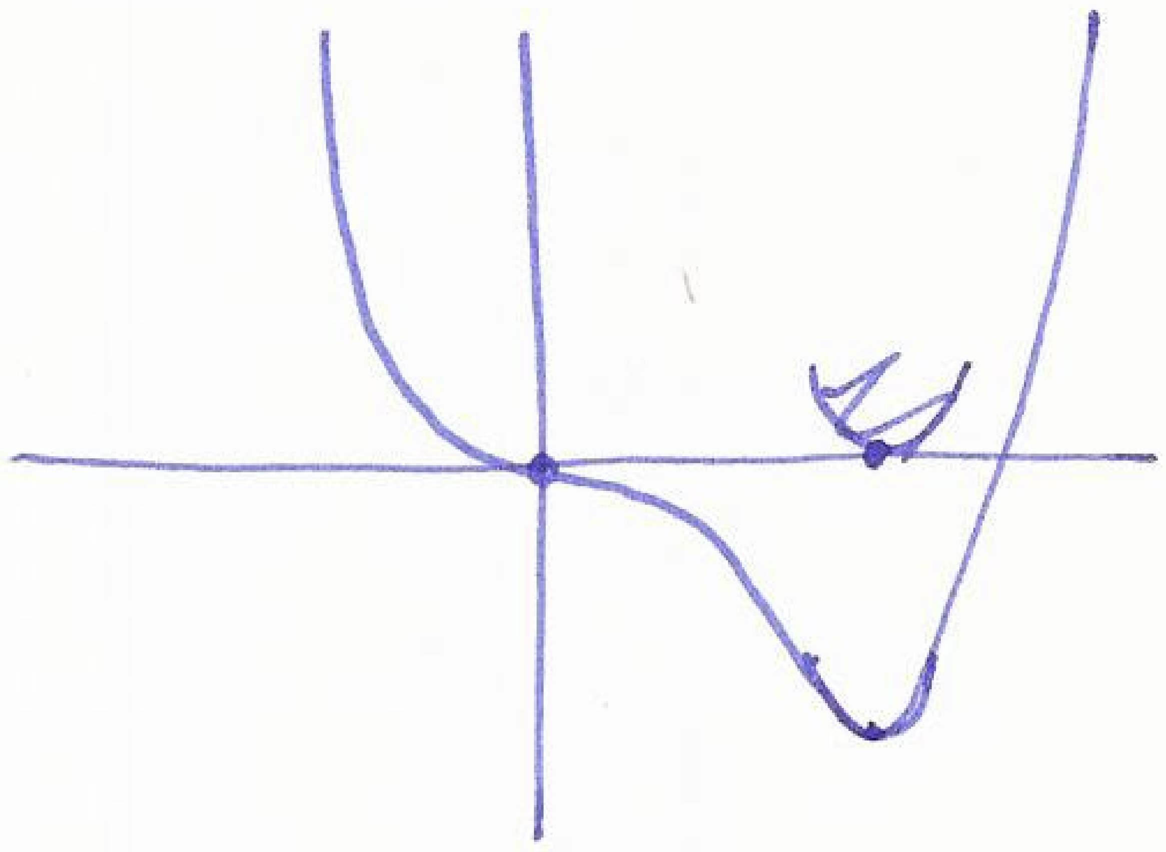
$(-\infty,0)$ $f'(x) < 0$ decreasing
$(0,3)$ $f'(x) < 0$ decreasing
$(3,\infty)$ $f'(x) > 0$ increasing

c) $g''(x) = 3x^2 - 6x = 3x(x-2)$ inflection points at $x = 0, 2$

d) concave up $g''(x) > 0$ $(-\infty, 0) \cup (2, \infty)$

down $g''(x) < 0$ $(0, 2)$

e)



Q5 a) vertical asymptotes: $x^2 - 4 = 0$ $(x-2)(x+2)$ as $x = \pm 2$

horizontal asymptotes: $\lim_{x \to \infty} \frac{1}{x^2-4} = 0$ $\lim_{x \to -\infty} \frac{1}{x^2-4} = 0.$

b) $f'(x) = -(x^2-4)^{-2} \cdot 2x = 0$ when $x = 0$.

c) $f'(x) = \frac{-2x}{(x^2-4)^2}$ can only change sign at $\pm 2, 0$

$(-\infty, -2)$ $f'(x) > 0$, $(-2, 0)$ $f'(x) > 0$ } increasing

$(0, 2)$ $f'(x) < 0$, $(2, \infty)$ $f'(x) < 0$ } decreasing

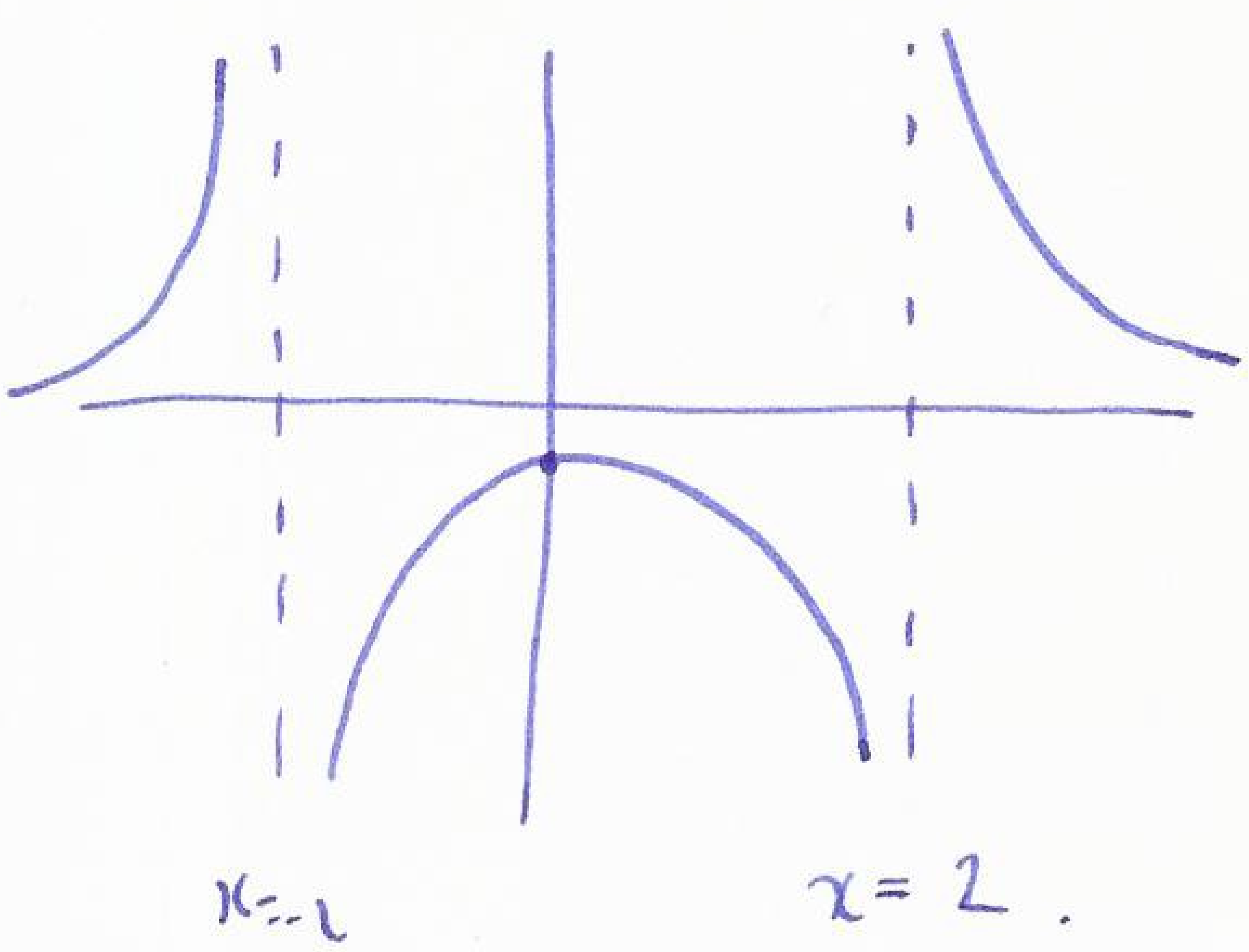
$f'(x) > 0$ increasing $f'(x) < 0$ decreasing

d) $f''(x) = \frac{(x^2-4)^2(-2) - \overbrace{2(x^2-4) \cdot 2x}(-2x)}{(x^2-4)^4}$ $= \cancel{\left(\frac{2x^4 + 8}{(x^2-4)^2} \right)}$ $\cancel{f''(0) = \frac{8}{(-4)^2} = \frac{1}{2}}$ local max.

$f''(0) = \frac{(-4)^2(-2) - 2(-4) \cdot 0 \cdot 0}{(0-4)^4}$.

$f''(0) < 0$ so local max.

e)



Q6 $f(x) = \sqrt[3]{x} = x^{1/3}$ linear approx at $x = 8$ with $\Delta x = 1$.

$$f'(x) = \frac{1}{3} x^{-2/3} \qquad f'(8) = \frac{1}{3}\frac{1}{4} = \frac{1}{12}$$

$$\Delta f \approx f'(8)\Delta x \approx \frac{1}{12}$$

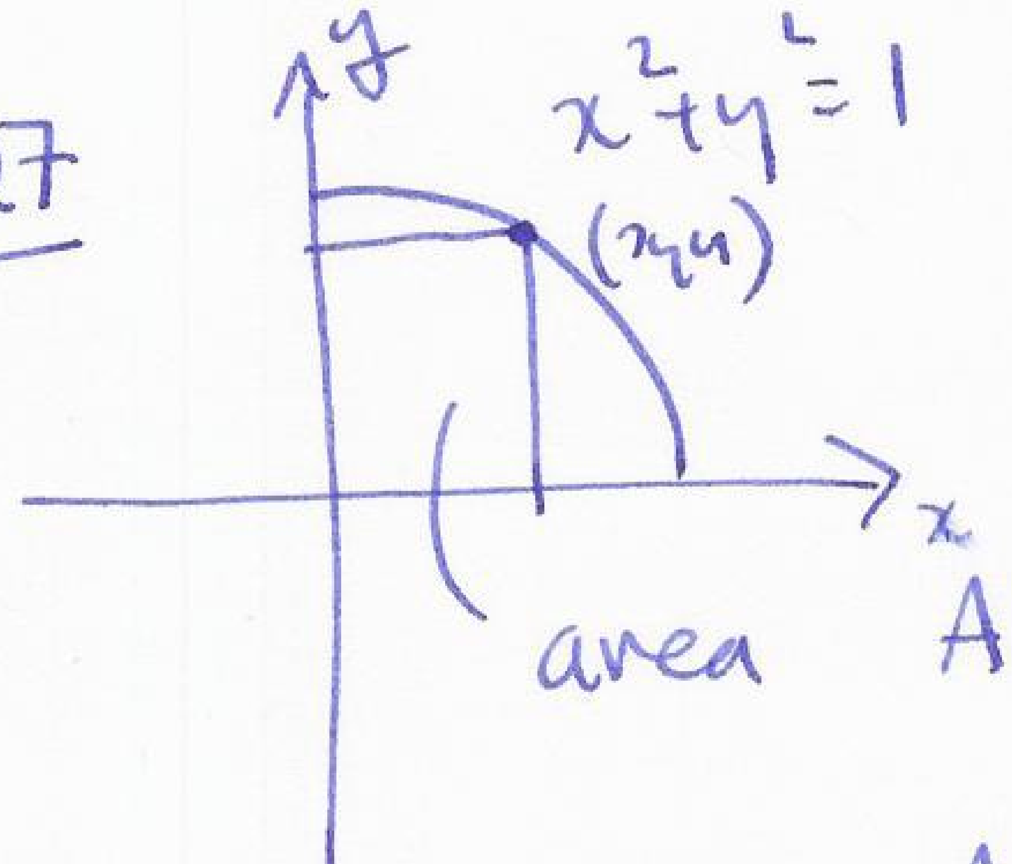
$$f(9) \approx \cancel{8}\,2 + \frac{1}{12} \approx 2.08333\ldots$$

$$\sqrt[3]{9} \approx 2.080083823\ldots$$

$$\text{error} \approx 0.0032495\ldots$$

$$\text{percentage error} = \frac{100 \times 0.0032495\ldots}{2.080083823\ldots} \approx 0.16\%.$$ good approximation

Q7



$x^2 + y^2 = 1 \qquad y = \sqrt{1-x^2}$.

area $A(x,y) = xy$

$$A(x) = x\sqrt{1-x^2}.$$

$$A'(x) = \sqrt{1-x^2} + x\,\frac{1}{2}(1-x^2)^{-1/2} \cdot -2x.$$

$$= \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}.$$

$A'(x) = 0 \iff x = \pm\frac{1}{\sqrt{2}}$ only +ve solution relevant.

so max area is $A\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$. (so in circle largest area rectangle is a square)

Q8 a) $$\lim_{x\to\infty} \frac{6x^5-12x^4-24}{2x^5+30} = \lim_{x\to\infty} \frac{6-12/x-24/x^5}{2+30/x^5} = 3.$$

b) $$\lim_{x\to-\infty} \frac{x^2}{1-x^2} = \lim_{x\to-\infty} \frac{1}{-1+1/x^2} = -1.$$

c) $$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+5}} = \cancel{\lim_{x\to\infty} \frac{3}{\frac{1}{2}(x^2+5)^{-1/2}\cdot 2x}} = \lim \frac{3}{\sqrt{1+5/x^2}} = 3.$$

d) $$\lim_{x\to 0} \frac{e^{2x^2}-1}{1-\cos(3x)} = \lim_{x\to 0} \frac{e^{2x^2}\cdot 4x}{3\sin(3x)} = \lim_{x\to 0} \frac{e^{2x^2}\cdot 4+e^{2x^2}\cdot 16x^2}{9\cos(3x)} = \frac{4}{9}$$