Common final solutions

Q1 a) $\dfrac{x+3}{x^2-9} = \dfrac{x+3}{(x+3)(x-3)} = \dfrac{1}{x-3} \quad (x \neq -3).$

so $\displaystyle\lim_{x\to 3} \frac{x+3}{x^2-9} = \lim_{x\to 3} \frac{1}{x-3}$ DNE as $\displaystyle\lim_{x\to 3+} \frac{1}{x-3} = +\infty$

$\displaystyle\lim_{x\to 3-} \frac{1}{x-3} = -\infty.$

b) $= \displaystyle\lim_{x\to 0} 4 \frac{\sin(4x)}{(4x)} = 4$ as $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1.$

c) (l'Hopital) $= \displaystyle\lim_{x\to 0} \frac{1}{\frac{1}{2}(x+9)^{-1/2}} = \lim_{x\to 0} 2\sqrt{x+9} = 6$

d) $= \displaystyle\lim_{x\to\infty} \frac{-3 + 2/x^2 - 10/x^4}{5 - 100/x^3 + 3/x^4} = -\frac{3}{5}$

Q2 $g(x)$ cts except possibly at $x=0$. $\quad g(0) = 5 - 3\cos(0) = 2$

$\displaystyle\lim_{g\to 0+} g(x) = b$ so want $b = 2$.

Q3 a) $f(x) = x^3 + e^{x\ln(3)} + e^x \quad f'(x) = 3x^2 + \ln(3)3^x + e^x.$

b) $f'(x) = \dfrac{(x^2+1)\cos x - (2x)\sin x}{(x^2+1)^2}$

c) $f'(x) = \cos(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = \dfrac{\cos(\sqrt{x})}{2\sqrt{x}}$

d) $f'(x) = 2x\ln(x) + x^2/x = 2x\ln(x) + x$

Q4 $3x^2 + 2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$

at $(2,1)$: $12 + 2\frac{dy}{dx} + 4\frac{dy}{dx} = 0$ $\quad \frac{dy}{dx} = -2$

tangent line: $y - 1 = -2(x-2)$ $\quad y = -2x + 5$.

Q5 a) $g'(x) = (x-4)^3 + x\,3(x-4)^2$

$g'(x) = 0$: $(x-4)^2\big((x-4) + 3x\big) = (x-4)^2 4(x-1) = 0$

$x = 4$, $x = 1$ critical points.

$$\begin{array}{ccccc} - & | & + & | & + \quad g' \\ & x=1 & & x=4 & \end{array}$$

b) increasing $\Leftrightarrow g'(x) > 0$ when $x > 1$

decreasing $\Leftrightarrow g'(x) < 0$ when $x < 1$

c) $x = 1$ local minimum $(1, -27)$

$x = 4$ neither max nor min. $(4, 0)$

d) $g''(x) = 8(x-4)(x-1) + 4(x-4)^2 = 4(x-4)(2x-2+x-4)$

$= 4(x-4)(3x-6) = 8(x-4)(x-2)$

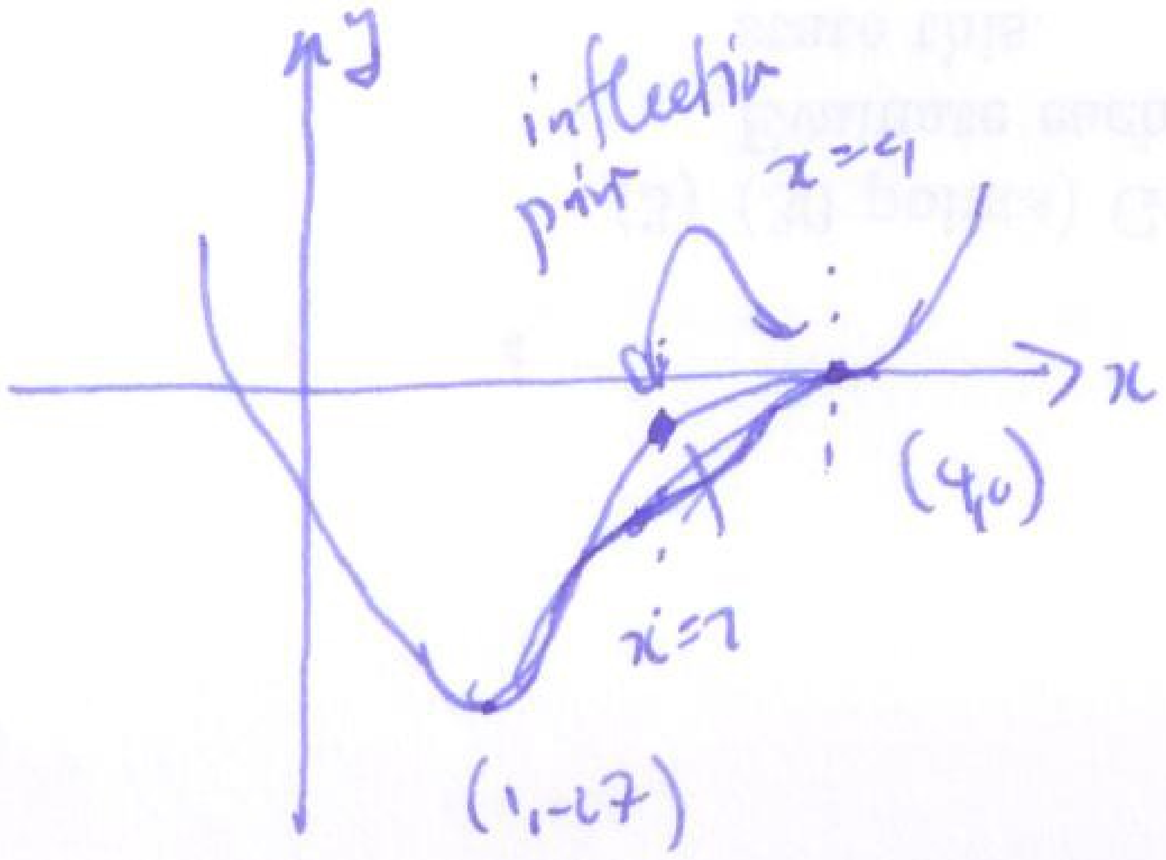
$$\begin{array}{cccccc} g'' & + & | & - & | & + \\ & & x=2 & & x=4 & \end{array}$$

points of inflection $x = 2, 4$

concave up $\Leftrightarrow g''(x) > 0$ when $x > 4$ $x < +2$

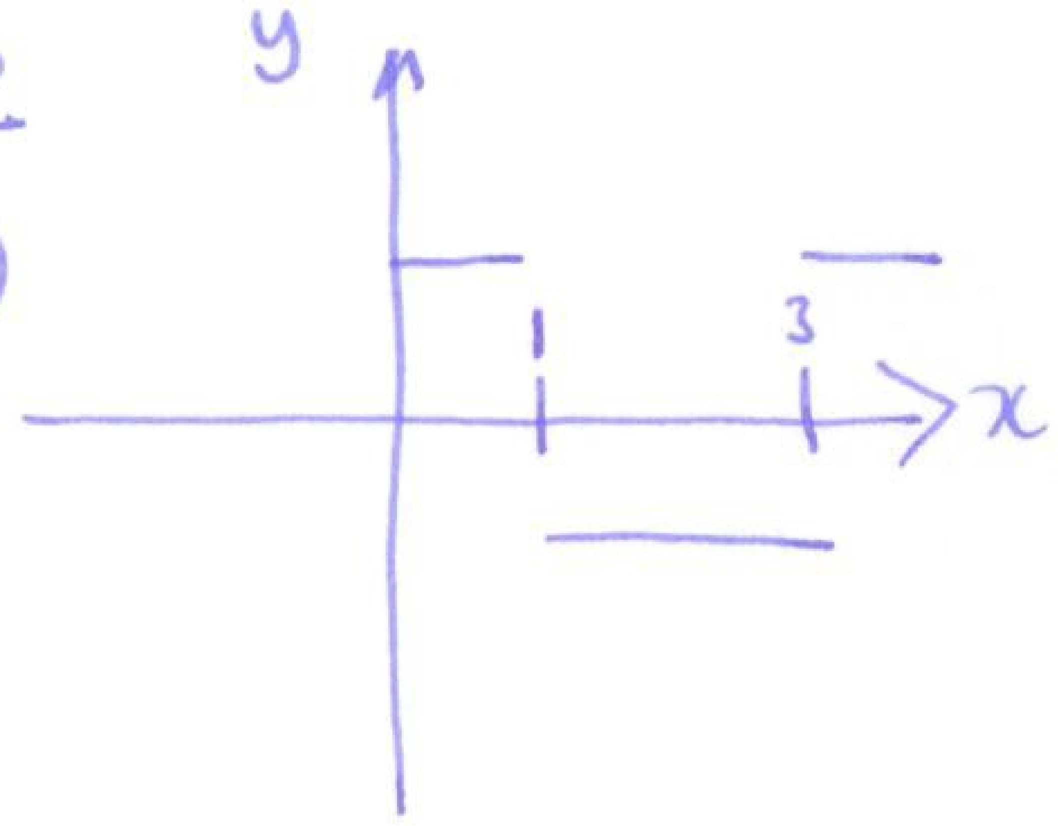
concave down $\Leftrightarrow g''(x) < 0$ when $2 < x < 4$

e)



Q6

a)



b) $\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$

c) $\frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 (2-x)\,dx\right]$

$= \frac{1}{2}\left(\left[\frac{1}{2}x^2\right]_0^1 + \left[2x - \frac{1}{2}x^2\right]_1^2\right) = \frac{1}{2}\left(\frac{1}{2} + 4 - 2 - 2 + \frac{1}{2}\right)$

$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$

Q7

a) $\left[\frac{1}{3}x^3 + \ln|x| + \frac{2x^{3/2}}{3}\right]_1^3 = 9 + \ln 3 + 2\sqrt{3} - \frac{1}{3} - 0 - \frac{2}{3}.$

$= 8 + \ln 3 + 2\sqrt{3}$

b) $\int e^x + 3 + e^{-x}\,dx = e^x + 3x - e^{-x} + c$

c) $\int_0^{\pi/4} \sin(2x)\,dx = \left[-\frac{1}{2}\cos(2x)\right]_0^{\pi/4} = -\frac{1}{2}\left(\cos(\pi/2) - \cos(0)\right) = -\frac{1}{2}(0-1)$

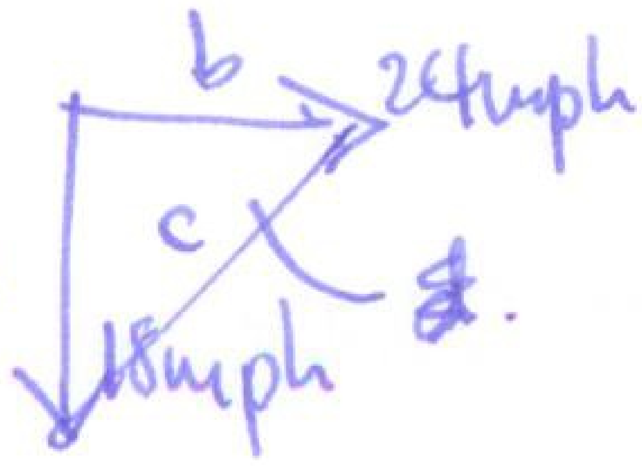
$= \frac{1}{2}.$

d) $\int_0^1 e^{x^3} x^2\,dx.$ sub $u = x^3$

$\frac{du}{dx} = 3x^2$

$\int_0^1 e^u x^2 \frac{1}{3x^2}\,du = \frac{1}{3}\int_0^1 e^u\,du = \frac{1}{3}\left[e^u\right]_0^1 = \frac{1}{3}(e-1)$

Q8

a) 

after 1/6 hours

Alex has travelled 3 miles

Brenda " " 4 miles.

$c\cancel{d}^2 = a^2+b^2 \qquad c\cancel{d} = \sqrt{a^2+b^2} \qquad \frac{dc}{dt} = \frac{1}{2}(a^2+b^2)^{-1/2}\left(2a\frac{da}{dt}+2b\frac{db}{dt}\right)$

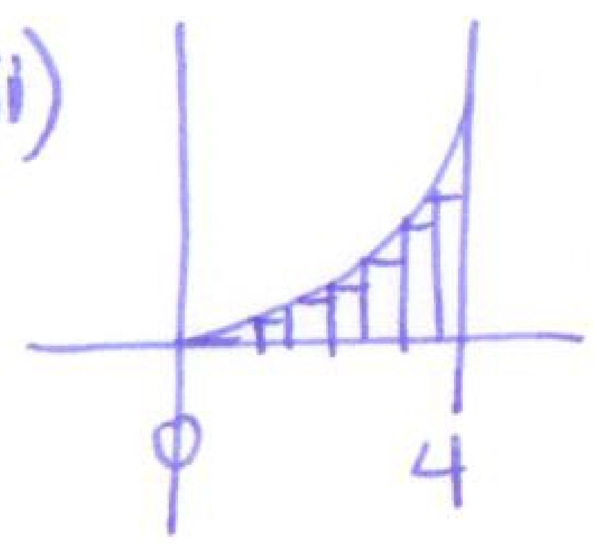
$a = 3 \quad \frac{da}{dt} = 18$

$b = 4 \quad \frac{db}{dt} = 24$

$\frac{dc}{dt} = \frac{1}{2}\frac{1}{\sqrt{3^2+4^2}}\left(6.18+8.24\right)$

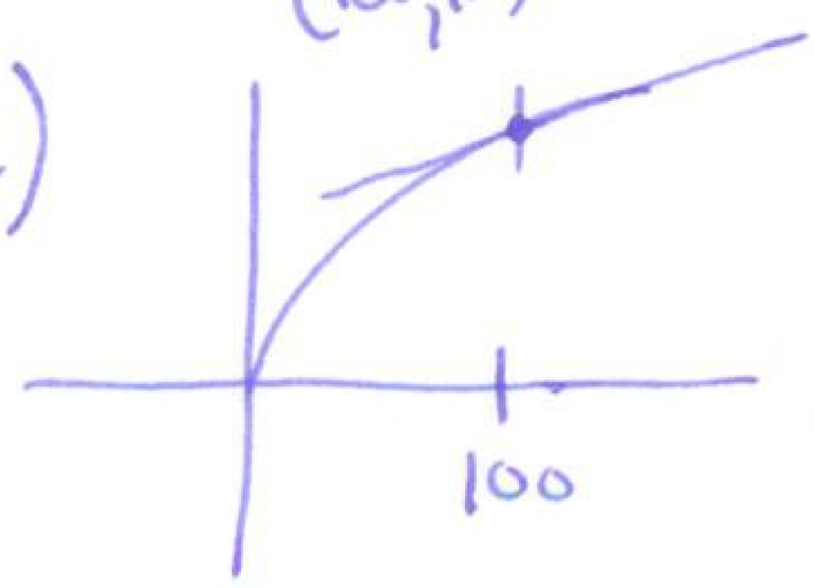
$= \frac{1}{10}\left(\overset{108}{\cancel{72}}+192\right) = 30$ mph.

b) i) $\int_0^4 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^4 = 4^3-0 = 64.$

ii) 

approximate area $= \sum_{i=0}^{n}\frac{4}{n}f\left(\frac{4i}{n}\right) = \sum_{i=0}^{n}\frac{4}{n}\frac{4^3 i^3}{n^3} = \frac{4^4}{n^4}\sum_{i=0}^{n} i^3$

$= \frac{4^4}{n^4}\frac{n^2(n+1)^2}{4} = 4^3(1+1/n)^2 \qquad \lim_{n\to\infty} 4^3(1+1/n)^2 = 64.$

c) 

$f(x) = \sqrt{x} \quad f'(x) = \frac{1}{2}x^{-1/2} \quad f'(100) = \frac{1}{20}$

$\Delta f \approx f'(100)\Delta x = \frac{1}{20}\cdot 2 = \frac{1}{10}$

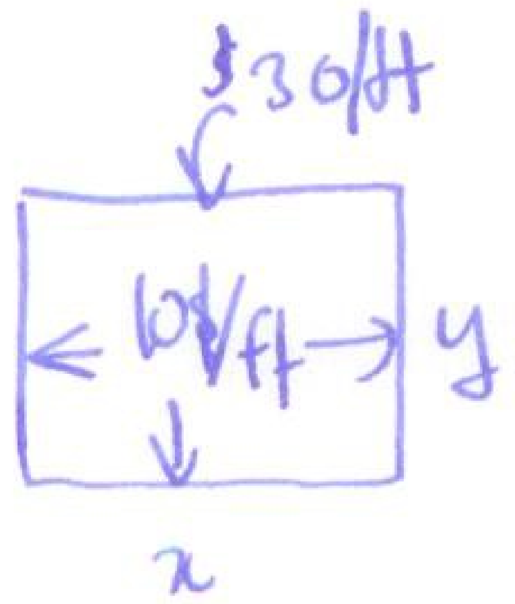
so $f(100+2) \approx f(100) + \frac{1}{10} = 10.1$

actual answer $\sqrt{102} = 10.09950494$ good approximation

error $= 0.000495\ldots$ percentage error $\approx 5\times10^{-3}$ (small)

5

d)



$$\text{cost} = 30x + 10x + 20y = 40x + 20y$$

$$xy = 1000 \quad \text{so} \quad y = \frac{1000}{x}.$$

$$\text{cost} = 40x + \frac{20000}{x} \qquad \frac{dc}{dx} = 40 + -\frac{20{,}000}{x^2} = 0.$$

$$x^2 = 500 \quad x = \sqrt{500} = 10\sqrt{5}, \; y = \frac{100}{\sqrt{5}}$$