§ 3.4 Differentials

1 var

\( y = f(x) \) the differential of \( y \) is \( dy = f'(x)\Delta x \)

\[ \Delta y = f(x + \Delta x) - f(x) \]
\[ \Delta y \approx dy = f'(x)\Delta x = f'(x)\Delta x. \]

\( \Delta x \): small change in \( x \).
\( \Delta y \): corresponding change in \( f(x) \).

\( dy = f'(x)dx \) a (linear) equation approximating \( \Delta y \) for small \( \Delta x \).

2 var

\[ \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \]

the differential of \( z \) is \( dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)dx + f_y(x, y)dy \)

(a linear equation approximating \( \Delta z \) for small \( \Delta x, \Delta y \)).

\[ \Delta z \approx f_x \Delta x + f_y \Delta y \]

3 vars \( \omega = f(x, y, z) \) has differential \( d\omega = f_x dx + f_y dy + f_z dz \)

\[ \Delta \omega \approx f_x \Delta x + f_y \Delta y + f_z \Delta z \] for small \( \Delta x, \Delta y, \Delta z \).

Example \( z = x^2 + y^2 \) \( dz = 2x dx + 2y dy \).

Differentiability of functions of two vars.

\( z = f(x, y) \) is differentiable at \( (x_0, y_0) \) if \( \Delta z \) can be written as

\[ \Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon(\Delta x, \Delta y) \quad \text{where} \quad \epsilon \to 0 \quad \text{as} \quad (\Delta x, \Delta y) \to 0. \]

Useful fact \( f_x f_y \) is at \( (x_0, y_0) \) \( \Rightarrow f \) is differentiable.
Using differentials for approximations

\[ z = f(x, y) : \Delta z \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y \]

**Example**  
\[ z = \sqrt{\frac{x^2}{6} - y^2} \]  
\[ f(x, y) = \frac{x^2}{6} - y^2 \]

Approximate \[ f(0.01, 0.01) \]
\[ f_x(x, y) = \frac{1}{2} \left( \frac{x^2}{6} - y^2 \right)^{-1/2} \cdot 2x \]
\[ f_y(x, y) = \frac{1}{2} \left( \frac{x^2}{6} - y^2 \right)^{-1/2} \cdot (-2y) \]

\[ \Delta z \approx f_x(1, 1)(0.01) + f_y(1, 1)(0.01) \]
\[ = \frac{-2}{2.2} (0.01) + \frac{-2}{2.2} (0.01) \approx -0.01 \]

\[ \therefore f(1.01, 1.01) \approx 1.99 \]

**Example**  
A tin can has a base \( 12 \text{ cm} \) in radius, \( 5 \text{ cm} \) high and walls \( 0.2 \text{ cm} \) in thickness, and the sides have thickness \( 0.1 \text{ cm} \). Estimate the amount of metal in the can.

Area \( \approx \) change in volume, \( V = 2\pi rh \)

\[ \Delta V \approx \frac{\partial V}{\partial r} \Delta r + \frac{\partial V}{\partial h} \Delta h = 2\pi rh \Delta r + 2\pi r h \Delta h \]

\[ h = 10, \Delta h = 0.2 \]
\[ r = 5, \Delta r = 0.1 \]

\[ \Delta V \approx \pi r h (0.1) + \pi r h (5 \times 0.2) \]
\[ \approx 6\pi \]
§B.5 Chain rule

Recall \[ \frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x). \]

For \( f(y), y = g(x) \), \[ \frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}. \]

Chain rule for many variables: “differentiate with respect to all possible variables.”

Special case \( w = f(x,y) \) where \( x = g(t) \)
\[ y = h(t) \]
\[ (w(t) = f(g(t),h(t)) \]
\[ \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \]

Example \( w = xy \) \[ x = t \]
\[ y = \sin(t) \]
\[ \frac{dw}{dt} = 2xy 1 + x^2 \cos(t) = 2t \sin(t) + t^2 \cos(t) \]

Special case \( w = f(x,y) \) \[ x = g(s,t) \]
\[ y = h(s,t) \]
\[ \frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \]
\[ \frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \]

Example \( w = xy \) \[ x = s^2 t^2 \]
\[ y = \sqrt{t} \]
\[ \frac{\partial w}{\partial s} = y 2s + x 2t = \frac{2s^2}{t} + 2t (s^2 t^2) \]