§ 13.1 Functions of several variables

We've seen functions of 1-variable: \( f: \mathbb{R} \rightarrow \mathbb{R} \) \( f: \mathbb{R} \rightarrow \mathbb{R}^2 \) etc.

In general: \( f: \text{(domain)} \rightarrow \text{(range)} \)
\[ x \mapsto f(x) \]
\[ t \mapsto (x(t), y(t)) \]

Functions of 2-variables:
\( f: \mathbb{R}^2 \rightarrow \mathbb{R} \)
\( (x, y) \mapsto f(x, y) \)

Example: \( (x, y) \mapsto x^2 y^2 \)

Convention: \( f: \text{domain} \rightarrow \text{range} \) (usually not defined explicitly).

Example: \( f: (x, y) \mapsto \frac{1}{x^2 y^2} \)

Domain is \( \mathbb{R}^2 \setminus (0,0) \).
Range is \( (0, \infty) \).

Example: \( \int_a^b f(x) \, dx \)

Function of \( a \) to \( b \)!

Not a function of \( f \).
Could be a function of \( f \) to \( \mathbb{R} \).

\[ \int_a^b \, dx : \{ \text{functions?} \} \rightarrow \mathbb{R} \]

Graphs of functions of two variables

1-variable:
\( f: \mathbb{R} \rightarrow \mathbb{R} \)
\( x \mapsto f(x) \)
\( x \mapsto x^2 \)

Graph of function is a curve.

2-variables:
\( f: \mathbb{R}^2 \rightarrow \mathbb{R} \)
\( (x, y) \mapsto f(x, y) \)
\( x \mapsto x^2 + y^2 \)

Graph of function is a surface.

Q: When is a surface a graph of a function? A: Vertical line test.

Sphere is not a graph of a function.
Level sets / contour lines

Q: how do we know what \( z = f(x,y) \) looks like?
A: by drawing level sets = contour lines = horizontal cross sections.

Example: \( f(x,y) = x^2 + y^2 \)

\[ z = f(x,y) = x^2 + y^2 \]

The graph needs an extra dimension.

Find intersection with horizontal plane \( z = c \) \( c = x^2 + y^2 \) circle of radius \( \sqrt{c} \).

Example: \( f(x,y) = x^2 - y^2 \). xy level sets in \( \mathbb{R} \).

\[ f: \mathbb{R}^2 \rightarrow \mathbb{R} \]

Functions of 3 variables

\( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) \( f(x,y,z) \) graph of function needs 4th.

Level sets: \( f(x,y,z) = c \) hence are surfaces in \( \mathbb{R}^3 \).

Examples: \( f(x,y,z) = x + y + z \) \( f(x,y,z) = x^2 + y^2 + z^2 \)