Math 233 Calculus 3 Fall 09 Midterm 3

Name: Solutions

(1) (a) Define the gradient vector and describe its geometric properties.
(b) Find the gradient vector for the function

$$f(x,y,z) = \frac{\sin(xy)}{y+z}$$

at the point $(\frac{\pi}{2}, 1, 1)$.

a) $\underline{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$\underline{\nabla} f$ points in the direction of fastest increase.

$\|\nabla f\|$ = rate of increase in that direction.

b) $\underline{\nabla} f = \langle \frac{y\cos(xy)}{y+z}, \frac{(y+z)x\cos(xy) - \sin(xy)}{(y+z)^2}, -\frac{\sin(xy)}{(y+z)^2} \rangle$

$\underline{\nabla} f(\frac{\pi}{2},1,1) = \langle 0, -\frac{1}{4}, -\frac{1}{4} \rangle$

(2) Find the tangent plane to the surface

$$\frac{\sin(xy)}{y+z} = \frac{1}{2}$$

at the point $(\frac{\pi}{2}, 1, 1)$.
Feel free to use your answer to the previous question.

normal vector is $\nabla F(\frac{\pi}{2}, 1, 1) = \langle 0, -\frac{1}{4}, -\frac{1}{4} \rangle$

equation of tangent plane:

$$0 \cdot \left(x - \frac{\pi}{2}\right) - \frac{1}{4}(y-1) - \frac{1}{4}(z-1) = 0$$

$$y - 1 + z - 1 = 0$$

$$y + z - 2 = 0.$$

(3) Find all first and second order partial derivatives of the function

$$f(x, y) = x^2 e^{y/2} + y e^{y/2}.$$

$\frac{\partial f}{\partial x} = 2x e^{y/2} + 0$

$\frac{\partial f}{\partial y} = x^2 \frac{1}{2} e^{y/2} + e^{y/2} + y \frac{1}{2} e^{y/2}$

$f_{xx} = 2e^{y/2}$

$f_{xy} = 2x \frac{1}{2} e^{y/2}$

$f_{yy} = x^2 \frac{1}{4} e^{y/2} + \frac{1}{2} e^{y/2} + \frac{1}{2} e^{y/2} + y \frac{1}{4} e^{y/2}$

(4) Find all the critical points of the function
$$f(x, y) = x^2e^{y/2} + ye^{y/2},$$
and use the second derivative test to attempt to classsify them. Feel free to use your answer to the previous question.

critical points, solve: $2xe^{y/2} = 0$

$$\frac{1}{2}x^2e^{y/2} + e^{y/2} + \frac{1}{2}ye^{y/2} = 0$$

w/e $e^{y/2} \neq 0$: $2x = 0 \Rightarrow x = 0$

$$\frac{1}{2}x^2 + 1 + \frac{1}{2}y = 0 \Rightarrow y = -2$$

one critical point at $(0,-2)$

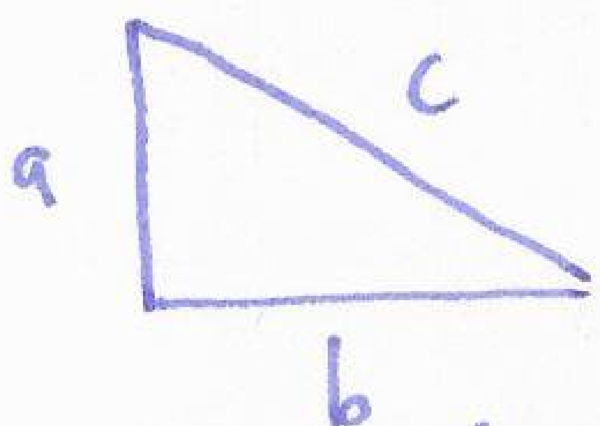
2nd derivative test: $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$

$$D(0,-2) = (2e^{-1}).(e^{-1} - \frac{1}{2}e^{-1}) - 0^2$$

$$= 2e^{-2} > 0$$

$f_{xx}(0,-2) = 2e^{-1} > 0$ so <u>local minimum</u>

(5) A right angled triangle has side lengths roughly 3m, 4m and 5m. You can measure any side to an accuracy of 0.1m. Use differentials to decide whether measuring the two longer sides gives you a more accurate estimate of the area than measuring the two shorter sides.



area $A = \frac{1}{2}ab = \frac{1}{2}\sqrt{c^2-b^2}\,b$

two short sides

$\Delta A \approx \frac{\partial A}{\partial a}\Delta a + \frac{\partial A}{\partial b}\Delta b$

$= \frac{1}{2}b\,\Delta a + \frac{1}{2}a\,\Delta b$

$= \frac{1}{2}4.0.1 + \frac{1}{2}.3.0.1$

$= 0.35 = \frac{7}{20} = \frac{21}{60}$

~~more accurate~~

two long sides

$\Delta A \approx \frac{\partial A}{\partial b}\Delta b + \frac{\partial A}{\partial c}\Delta c$

$\left[\frac{1}{2}\cdot\frac{1}{2}(c^2-b^2)^{-1/2}.-2b.b + \frac{1}{2}\sqrt{c^2-b^2}\right]\Delta b$

$+ \frac{1}{2}\cdot\frac{1}{2}(c^2-b^2)^{-1/2}.2c.b\,\Delta c$

$\left[-\frac{1}{2}\frac{1}{3}.4^2. + \frac{1}{2}3\right]0.1$

$+ \frac{1}{2}\frac{1}{3}.4.5.0.1.$

$\left[\frac{3}{2} - \frac{8}{3} + \frac{10}{3}\right]0.1$

$\left[\frac{3}{2} + \frac{2}{3}\right]0.1 = \frac{13}{6}.0.1 = \frac{13}{60}$

< 0.35

two longer sides more accurate.

(6) $f(x, y)$ is a function of x and y, and both $x(u, v)$ and $y(u, v)$ are functions of u and v. Suppose you know that

$$
\begin{array}{rlrl}
x(1,1) = 2, & y(1,1) = 3, & x_u(1,1) = 1, & y_u(1,1) = -1, \\
f_x(1,1) = -1, & f_y(1,1) = -2, & f_x(2,3) = 4, & f_y(2,3) = -4. \\
f_x(1,-1) = -2, & f_y(1,-1) = 2. & &
\end{array}
$$

(a) Compute $f_u(1, 1)$, or state what extra information you need to compute it.

(b) Compute $f_v(1, 1)$, or state what extra information you need to compute it.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial u}(1,1) = \frac{\partial f}{\partial x}(2,3) \cdot \frac{\partial x}{\partial u}(1,1) + \frac{\partial f}{\partial y}(2,3)\frac{\partial y}{\partial u}(1,1)$$

$$= 4 . 1 + (-4).(-1) = 8$$

$$\frac{\partial f}{\partial v}(1,1) = \underbrace{\frac{\partial f}{\partial x}(2,3)}_{4} \frac{\partial x}{\partial v}(1,1) + \underbrace{\frac{\partial f}{\partial y}(2,3)}_{-4} \frac{\partial y}{\partial v}(1,1)$$

need to know $\frac{\partial x}{\partial v}(1,1)$ and $\frac{\partial y}{\partial v}(1,1)$.