Sample midterm 3 Solutions

Q1

$$\left.\begin{array}{l} \lim\limits_{(t,0)\to(0,0)} \dfrac{0}{t^2+0} = 0 \\[2em] \lim\limits_{(t,t)\to(0,0)} \dfrac{t^2}{t^2+t^2} = \dfrac{1}{2} \end{array}\right\}$$ therefore $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Q2 $f(x,y) = x\cos(y+2x)$

$$f_x = \cos(y+2x) - x\sin(y+2x).2$$

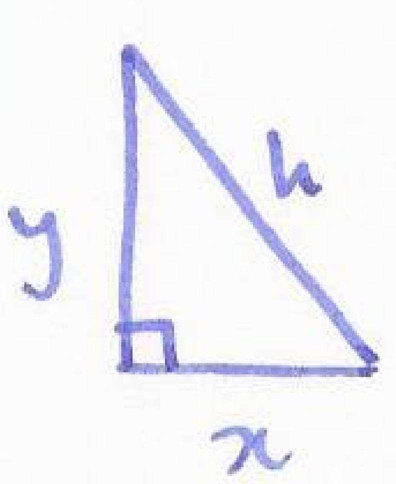
$$f_y = -x\sin(y+2x)$$

$$f_{xx} = -\sin(y+2x).2 - 2\sin(y+2x) - 2x\cos(y+2x).2$$

$$f_{xy} = -\sin(y+2x) - 2x\cos(y+2x)$$

$$f_{yy} = -x\cos(y+2x)$$

Q3



area $= \frac{1}{2}xy$ $\qquad \Delta A \approx \frac{1}{2}y\Delta x + \frac{1}{2}x\Delta y$

$h = \sqrt{x^2+y^2}$ $\qquad \Delta h \approx \frac{1}{2}(x^2+y^2)^{-1/2}.2x\Delta x + \frac{1}{2}(x^2+y^2)^{-1/2}.2y\Delta y$

$$\Delta A \approx \frac{1}{2}12.\frac{0.2}{100} + \frac{1}{2}5.\frac{0.2}{100} = \frac{1.7}{100} = 0.017$$

$$\Delta h \simeq \frac{1}{2}(25+144)^{-1/2}\, 2.5.\frac{0.2}{100} + \frac{1}{2}(25+144)^{-1/2}.2.12.\frac{0.2}{100}$$

$$\frac{1}{13.100} + \frac{2.4}{13.100} = \frac{3.4}{13.100}.$$

Q4 $\quad z = \cos(xy) + y\cos(x) \qquad x = u^2+v \qquad y = u - v^2$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \left(-y\sin(xy).\cancel{2u} + -y\cos(x)\right).2u + \left(-x\sin(xy) + \cos(x)\right).1$$

$$= -2u(u-v^2)\left(\sin\left(\cancel{u^4 v^3}\ (u^2+v)(u-v^2)\right) + \cos(u^2+v)\right) - (u^2+v)\sin\left(\cancel{u^4 - v^2}\ (u^2+v)(u-v^2)\right) + \cos(u^2+v)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \left(-y\sin(xy) - y\cos(x)\right).1 + \left(-x\sin(xy) + \cos(x)\right).-2v$$

$$= -(u-v^2)\left(\sin\left((u^2+v)(u-v^2)\right) - \cos(u^2+v)\right)$$

$$\cancel{+2u} -2v\left(-(u^2+v)\sin\left((u^2+v)(u-v^2)\right) + \cos(u^2+v)\right)$$

Q5 $f(x,y,z) = \tan(xz) + e^{xyz}$

a) $\underline{\nabla} f$ is a vector which points in the direction of fastest increase.

$\|\underline{\nabla} f\|$ is the rate of increase in that direction.

b) $\underline{\nabla} f = \langle \sec^2(xz) \cdot z + yze^{xyz},\ xze^{xyz},\ \sec^2(xz) \cdot x + xye^{xyz} \rangle$

$\underline{\nabla} f(1,0,-1) = \langle -\sec^2(-1),\ -1,\ \sec^2(-1) \rangle$

Q6 $F(x,y,z) = x^3 + y^3 + z^3 = 24$

$\underline{\nabla} F = \langle 3x^2, 3y^2, 3z^2 \rangle$

$\underline{\nabla} F(2,2,2) = \langle 24, 24, 24 \rangle$ parallel to $\langle 1,1,1 \rangle$

tangent plane: $(x-2) + (y-2) + (z-2) = 0$

$x + y + z = 8.$

Q7 $f(x,y) = x + xy + \dfrac{1}{x+y}$

$$\left.\begin{aligned} f_x &= 1 + y - \frac{1}{(x+y)^2} = 0 \quad ① \\ f_y &= x - \frac{1}{(x+y)^2} = 0 \quad ② \end{aligned}\right\} \quad ① - ② : \ 1 + y - x = 0 \iff x = y + 1$$

sub into ①: $1 + y - \dfrac{1}{(2y+1)^2} = 0$

$(1+y)(1+2y)^2 = 1$

$$(1+y)(1+4y+4y^2) = 1$$

$$1 + 4y + 4y^2$$
$$y + 4y^2 + 4y^3 = 1$$

$$5y + 8y^2 + 4y^3 = 0$$

$$y(5 + 8y + 4y^2) = 0$$

$$y((2y+2)^2 + 1) = 0$$

↑ no solutions, so only solution is $y = 0$.

$y=0$: ①: $1 - \frac{1}{x^2} = 0$ $x^2 = 1$ $x = \pm 1$

②: $x - \frac{1}{x^2} = 0$ $x^3 = 1$ $x = +1$

so only solution is $(1,0)$

$$f_{xx} = \frac{2}{(x+y)^3}$$

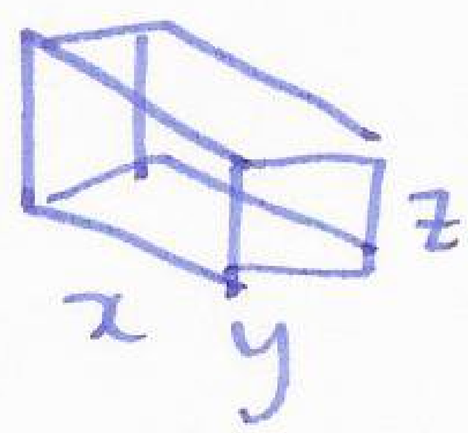
$$f_{xy} = 1 + \frac{2}{(x+y)^3}$$

$$f_{yy} = \frac{2}{(x+y)^3}$$

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

$$D(1,0) = 2.2 - (1+2)^2 = -5, \underline{\text{saddle}}.$$

<u>Q8</u>



max $V = xyz$

subject to: $4x + 4y + 4z = 64$

$$z = 16 - x - y$$

$$\max \quad V = xy(16-x-y) = 16xy - x^2y - xy^2$$

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= 16y - 2xy - y^2 = 0 \\ \frac{\partial V}{\partial y} &= 16x - x^2 - 2xy = 0 \end{aligned}\right\} \quad \begin{aligned} y(16 - 2x - y) &= 0 \\ x(16 - x - 2y) &= 0 \end{aligned}$$

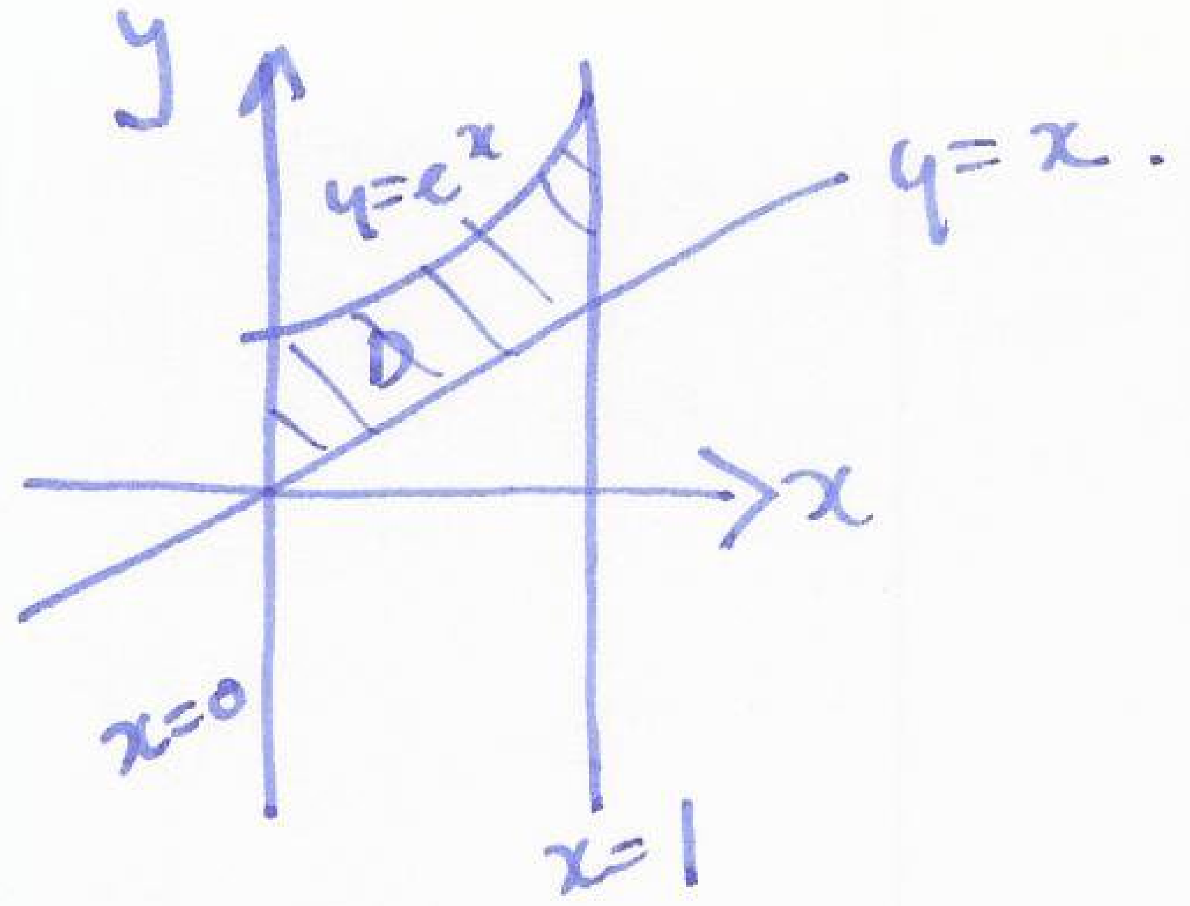
$x \neq 0, y \neq 0$ at max :

$$\left.\begin{aligned} 16 - 2x - y &= 0 \quad (1) \\ 16 - x - 2y &= 0 \quad (2) \end{aligned}\right\}$$

$(1) - 2(2)$: $-16 + 3y = 0 \qquad y = 16/3 \Rightarrow x = 16/3, \ z = 16/3$

so $V = \left(\frac{16}{3}\right)^3$.

Q9



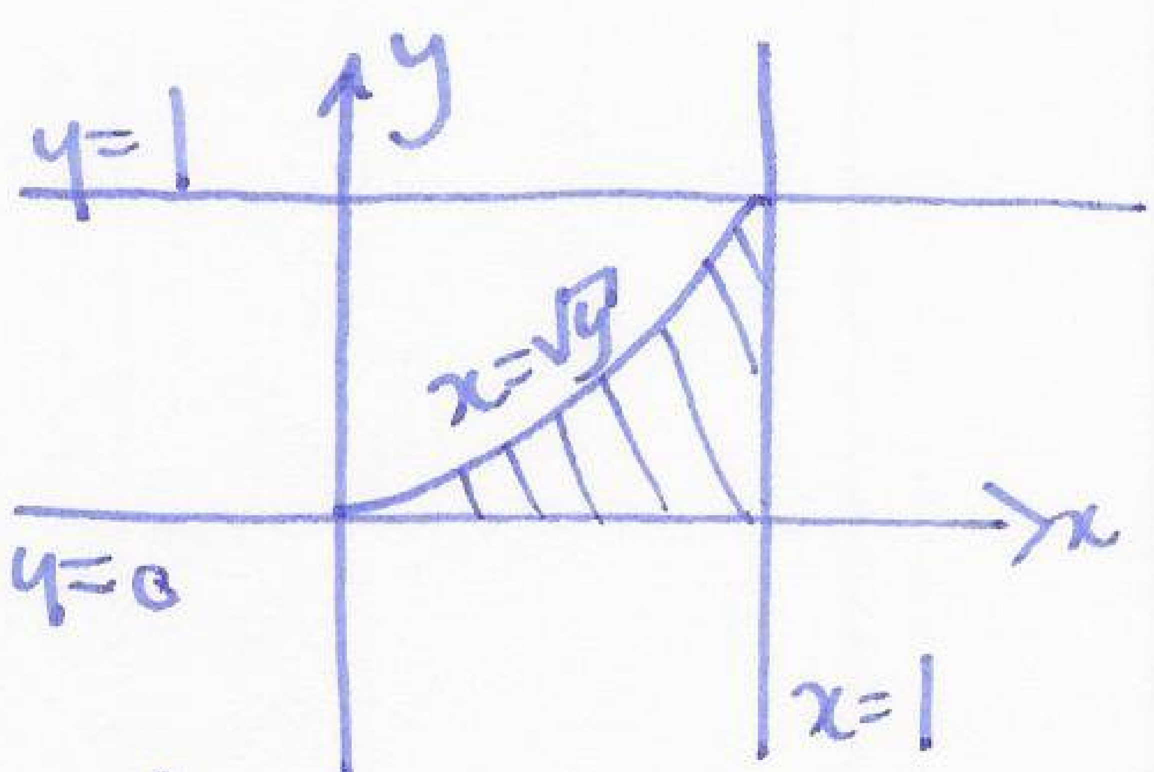
$$\int_0^1 \int_x^{e^x} 3xy^2 \, dy \, dx$$

$$\left[xy^3 \right]_x^{e^x} = xe^{3x} - x^4$$

$$\int_0^1 xe^{3x} - x^4 \, dx = \left[x\tfrac{1}{3}e^{3x} \right]_0^1 - \int_0^1 \tfrac{1}{3}e^{3x} dx - \left[\tfrac{1}{5}x^5 \right]_0^1 .$$

$$= \tfrac{1}{3}e^3 - \left[\tfrac{1}{9}e^{3x} \right]_0^1 - \tfrac{1}{5} = \tfrac{1}{3}e^3 - \tfrac{1}{9} - \tfrac{1}{5} .$$

Q10



$x = \sqrt{y}$ $\quad x^2 = y$

$$\int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3} \, dy \, dx \qquad \left[\tfrac{1}{2}y^2 \frac{e^{x^2}}{x^3} \right]_0^{x^2} = \tfrac{1}{2}xe^{x^2}$$

$$\int_0^1 \tfrac{1}{2}xe^{x^2} dx = \left[\tfrac{1}{4}e^{x^2} \right]_0^1 = \tfrac{1}{4}(e-1).$$