Math 233 Calculus 3 Fall 09 Midterm 2

Name: Solutions

(1) Find equations for the surface $z = xy$ in both cylindrical and spherical coordinates.

cylindrical:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$z = xy$

$\Leftrightarrow$

$z = r\cos\theta\, r\sin\theta$

$z = r^2\sin\theta\cos\theta$.

spherical:

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$z = xy$

$\Leftrightarrow$

$\rho\cos\phi = \rho^2\sin^2\phi\sin\theta\cos\theta$

$\cos\phi = \rho\sin^2\phi\sin\theta\cos\theta$

(2) A particle starts at time $t = 0$ the point $\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ with velocity $-\boldsymbol{j} - \boldsymbol{k}$, and moves with acceleration $\boldsymbol{a}(t) = \langle 4e^{2t}, 6t, \sin(t)\rangle$. Find the motion of the particle.

$\underline{r}''(t) = \langle 4e^{2t}, 6t, \sin t \rangle$

$\underline{r}'(t) = \langle 2e^{2t}, 3t^2, -\cos t \rangle + \underline{c}$

$\underline{r}'(0) = \langle 2, 0, -1 \rangle + \underline{c} = \langle 0, -1, -1 \rangle$

so $\underline{c} = \langle -2, -1, 0 \rangle$

$\underline{r}'(t) = \langle 2e^{2t} - 2, 3t^2 - 1, -\cos t \rangle$

$\underline{r}(t) = \langle e^{2t} - 2t, t^3 - t, -\sin t \rangle + \underline{c}$

$\underline{r}(0) = \langle 1, 0, 0 \rangle + \underline{c} = \langle 1, 2, 3 \rangle$

so $\underline{c} = \langle 0, 2, 3 \rangle$

$\underline{r}(t) = \langle e^{2t} - 2t, t^3 - t + 2, -\sin t + 3 \rangle$

(3) The position of a particle is given by $\boldsymbol{r}(t) = \langle \frac{1}{2}t^2, \sqrt{2}t, \ln t \rangle$. Find the length of the path the particle follows between $t = 1$ and $t = 4$.

$\underline{r}'(t) = \langle t, \sqrt{2}, \frac{1}{t} \rangle$

$\|\underline{r}'(t)\| = \sqrt{t^2 + 2 + \frac{1}{t^2}} = \sqrt{(t+\frac{1}{t})^2} = t + \frac{1}{t}$

arc length $= \int_1^4 t + \frac{1}{t} \, dt = \left[\frac{1}{2}t^2 + \ln t \right]_1^4 = 8 + \ln 4 - \frac{1}{2}$

$= 7\frac{1}{2} + \ln 4$

(4) (a) What is the difference between speed and velocity?
(b) If a particle moves with constant speed does it have to move in a straight line?

Suppose that a parameterized curve $\boldsymbol{r}(t)$ is parameterized by arc length.

(c) What does that tell you about $\boldsymbol{r}'(t)$?
(d) Show that $\boldsymbol{r}'(t)$ is perpendicular to $\boldsymbol{r}''(t)$.

a) velocity is a vector giving rate of change of position.
speed is the length of the velocity vector

b) no.

c) $\|\underline{r}'(t)\| = 1$

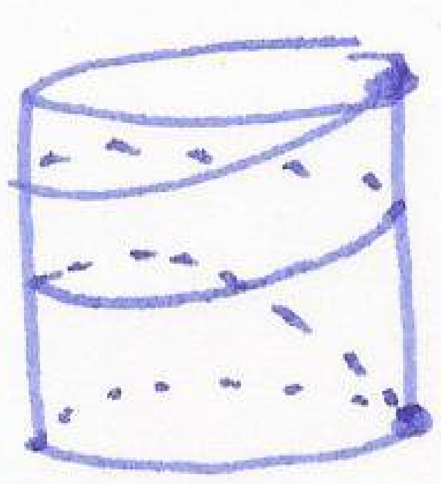
d) $\underline{r}'(t) . \underline{r}'(t) = 1$

differentiate: $2\underline{r}'(t) . \underline{r}''(t) = 0$

$\Rightarrow \underline{r}'(t) . \underline{r}''(t) = 0$

so $\underline{r}'(t)$ and $\underline{r}''(t)$ are perpendicular.

(5) A fairground slide is build around the side of a cylindrical tower 10 metres tall, with diameter of 10 metres. The top of the slide is immediately above the bottom of the slide, and you go round the building twice on the way down, with constant slope in the vertical direction.
  (a) Write down a parameterization for the slide.
  (b) Find the length of the slide.



a) $\underline{r}(t) = < 5\cos 2\pi t, 5\sin 2\pi t, \frac{10t}{2} >$

$0 \le t \le 2$

$\underline{r}'(t) = < -10\pi \sin 2\pi t, 10\pi \cos 2\pi t, 5 >$

$\|\underline{r}'(t)\| = \sqrt{100\pi^2 \sin^2 2\pi t + 100\pi^2 \cos^2 2\pi t + 25}$

$= \sqrt{100\pi^2 + 25}$

arc length $= \int_0^2 \sqrt{100\pi^2+25}\, dt = \left[\sqrt{100\pi^2+25}\, t \right]_0^2$

$= 2\sqrt{100\pi^2+25}$

(6) Write down the equation for the level sets of the function $f(x, y) = 1/(x^2 + y^2 + 1)$, and sketch some of them.

level sets : $\frac{1}{x^2+y^2+1} = c$

$\frac{1}{c} = x^2+y^2+1$

$x^2+y^2 = \frac{1}{c} - 1$