Math 233 Calculus 3 Fall 09 Final

Name: Solutions

(1) Let $\mathbf{v}$ be the vector $\langle 1, -1, 1\rangle$, and let $\mathbf{w}$ be the vector $\langle 3, 2, 2\rangle$.
   (a) Write $\mathbf{w}$ as a sum of two vectors, one parallel to $\mathbf{v}$, and one perpendicular to $\mathbf{v}$.
   (b) Find the equation of the plane through the origin which contains the two vectors $\mathbf{v}$ and $\mathbf{w}$.

a) find projection of $\underline{w}$ onto $\underline{v}$: $(\underline{v}.\underline{w})\dfrac{\underline{v}}{\|\underline{v}\|^2} = (3-2+2)\dfrac{<1,-1,1>}{3}$

$= <1,-1,1>$

so $\underline{w} = <1,-1,1> + <2,3,1>$

b) find a perpendicular vector : $\underline{v} \times \underline{w}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 3 & 2 & 2 \end{vmatrix} = <-4, 1, 5>$$

equation of plane : $-4x + y + 5z = 0$

(2) (a) What is the difference between speed and velocity?

(b) A particle starts at the origin at time zero, and has velocity given by $\mathbf{r}'(t) = \langle 3\sin(t), -4\sin(t), 5\cos(t)\rangle$. Where is the particle at time $t = \pi$?

a) velocity is a vector giving the rate of change of position with time. Speed is the length of the velocity vector.

b) $\underline{r}(t) = \langle -3\cos(t), 4\cos(t), 5\sin(t)\rangle + \underline{c}$

$\underline{r}(0) = \langle -3, 4, 0\rangle + \underline{c} = \langle 0, 0, 0\rangle \Rightarrow \underline{c} = \langle 3, -4, 0\rangle$

$\underline{r}(\pi) = \langle -3\cos(\pi), 4\cos(\pi), 5\sin(\pi)\rangle + \langle 3, -4, 0\rangle$

$= \langle 3, -4, 0\rangle + \langle 3, -4, 0\rangle = \langle 6, -8, 0\rangle.$

(3) (a) Define the gradient vector and describe its geometric properties.
(b) Find the gradient vector of the function $f(x, y) = x^2 \sin(3x + 2y)$ at the point $(2, -3)$.

$\nabla f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

a) The gradient vector points in the direction of fastest rate of change, and its length is the fastest rate of change.

b) $\nabla f = \langle 2x \sin(3x+2y) + x^2 3\cos(3x+2y),\ 2x^2 \cos(3x+2y) \rangle$

$\nabla f(2,-3) = \langle 4\sin(0) + 12\cos(0),\ 8\cos(0) \rangle = \langle 12, 8 \rangle.$

(4) Find all second partial derivatives of the function $f(x, y) = x^2 - y^3 + 4x + 6y^2 - 10$.

$f_x = 2x + 4$

$f_y = -3y^2 + 12y$

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = -6y + 12$

(5) Find the critical points of the function $f(x, y) = x^2 - y^3 + 4x + 6y^2 - 10$ and use the second derivative test to classify them, if possible. Feel free to use your answer to the previous question.

Solve $\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\}$ $\quad 2x + 4 = 0 \quad x = -2$

$-3y^2 + 12y = 0 \quad 3y(-y+4) = 0 \quad y = 0, 4$

two critical points $(-2, 0)$ and $(-2, 4)$

2nd derivative test: $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

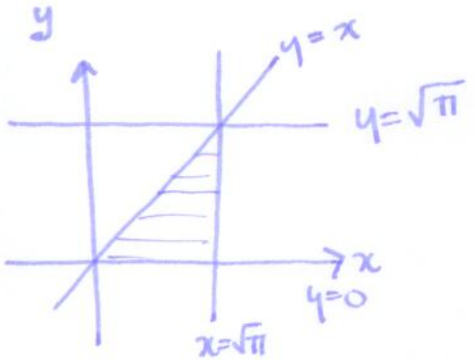
$D(-2,0) = 2(12) - 0^2 = 24 > 0 \quad f_{xx} = 2 > 0$

so local min.

$D(-2,4) = 2(-12) - 0^2 = -24 < 0$ so saddle point.

(6) Evaluate the following double integral by changing the order of integration.

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(\tfrac{1}{2}x^2)\,dx\,dy$$



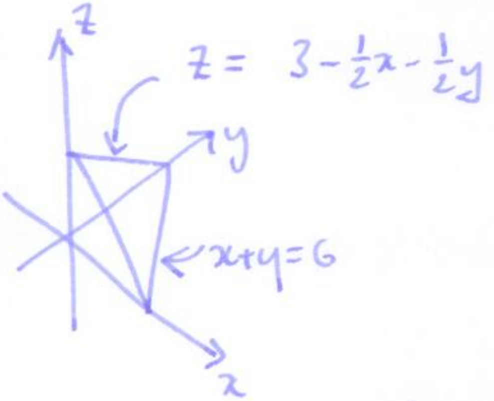
$$\int_0^{\sqrt{\pi}} \int_0^{x} \sin\left(\tfrac{1}{2}x^2\right) dy\, dx$$

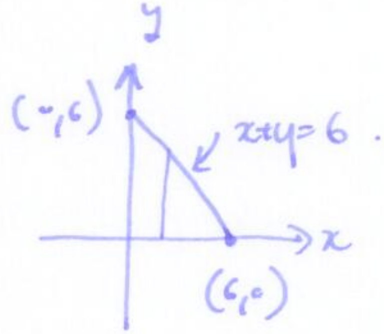
$$\left[y \sin\left(\tfrac{1}{2}x^2\right) \right]_0^x = x \sin\left(\tfrac{1}{2}x^2\right)$$

$$\int_0^{\sqrt{\pi}} x \sin\left(\tfrac{1}{2}x^2\right) dx = \left[-\cos\left(\tfrac{1}{2}x^2\right) \right]_0^{\sqrt{\pi}}$$

$$= -\cos\left(\tfrac{\pi}{2}\right) + \cos(0) = 1$$

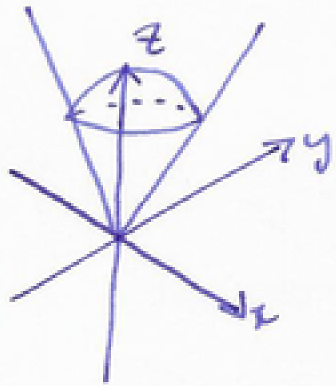
(7) Write down a triple integral which gives the integral of the function $f(x, y, z) = xyz$ over the region in the positive octant underneath the plane $x+y+2z=6$ using a triple integral. **Do not evaluate this integral.**





$$\int_0^6 \int_0^{6-x} \int_0^{3-\frac{1}{2}x-\frac{1}{2}y} xyz \; dz\, dy\, dx$$

(8) Write down a triple integral which gives the integral of the function $f(x, y, z) = x + y + z$ over the ice cream cone shaped region above the positive cone $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = 1$. **Do not evaluate this integral.**



use spherical coordinates

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$x^2+y^2+z^2 \Leftrightarrow \rho^2 = 1$

$z^2 = x^2+y^2$

$\Updownarrow$

$\rho^2 \cos\phi = \rho^2 \sin\phi (\cos^2\theta + \sin^2\theta)$

$\Updownarrow \tan\phi = 1$

$\phi = \pi/4$

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/4} (\rho \sin\phi (\cos\theta + \sin\theta) + \rho\cos\phi)\, \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$