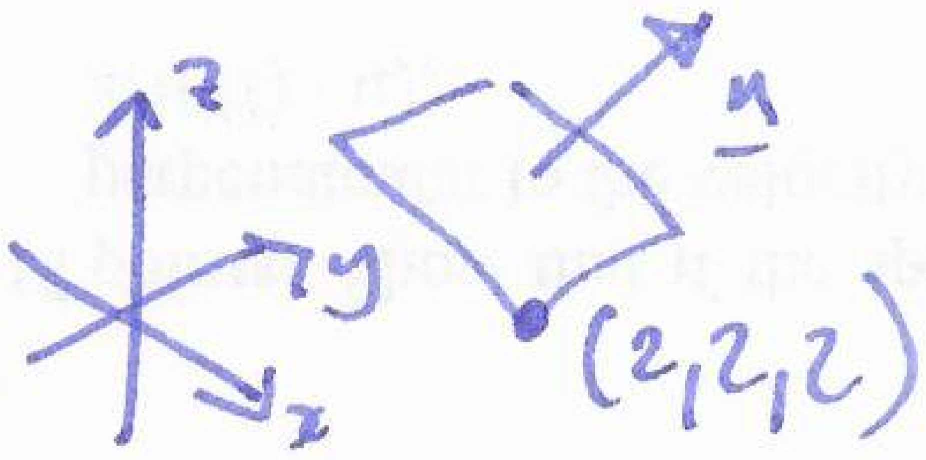
Sample final solutions

Q1 a) $\underline{n} = \langle 1,1,-1\rangle \times \langle 1,0,0\rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$

$= \langle 0,-1,-1\rangle$

b) $0(x-2) - (y-2) - (z-2) = 0$

Q2 a) 

distance to origin = length of projection of $\langle 2,2,2\rangle - \langle 0,4,0\rangle$ onto $\langle 0,1,1\rangle$

$= \langle 2,2,2\rangle \cdot \dfrac{\langle 0,1,1\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\cdot 4 = 2\sqrt{2}$

b) distance from origin $d^2 = \underbrace{x^2+y^2+z^2}_{F}$ minimize this, subject to

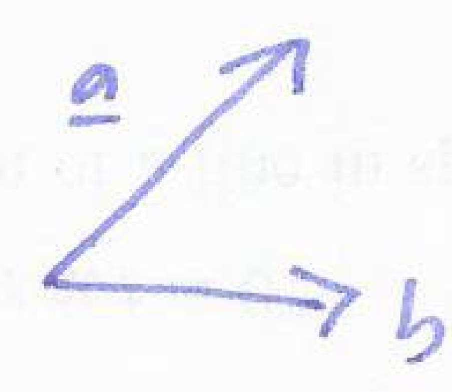
$\underbrace{y+z}_{G} = 4$ $\quad \nabla F = \langle 2x, 2y, 2z\rangle \quad \nabla G = \langle 0,1,1\rangle$

$\nabla F = \lambda \nabla G: \quad \left.\begin{aligned} 2x &= 0 \\ 2y &= \lambda \\ 2z &= \lambda \\ y+z &= 4 \end{aligned}\right\}$ $\quad x=0,\ y=2,\ z=2$

distance $= \sqrt{4+4} = 2\sqrt{2}$

Q3 $\underline{a} = \langle -2,4,2\rangle$

$\underline{b} = \langle 1,0,-1\rangle$



projection of $\underline{a}$ to $\underline{b}$: $\dfrac{\underline{a}\cdot\underline{b}}{|\underline{b}|^2}\,\underline{b} = \dfrac{(-2-2)}{2}\langle 1,0,-1\rangle$

$= \langle -2,0,2\rangle$

so $\underline{a} = \langle -2,0,2\rangle + \langle 0,4,0\rangle$

2

Q4 $\underline{x}(t) = \langle \cos t + t\sin t,\ \sin t - t\cos t,\ t^2 \rangle$

arc length $= \int_0^{\pi/2} \|\underline{x}'(t)\|\, dt$

$\underline{x}'(t) = \langle -\sin t + \sin t + t\cos t,\ \cos t - \cos t + t\sin t,\ 2t \rangle$

$$\int_0^{\pi/2} \sqrt{t^2\cos^2 t + t^2\sin^2 t + 4t^2}\, dt = \int_0^{\pi/2} \sqrt{5}\, t\, dt = \left[\frac{\sqrt{5}}{2} t^2 \right]_0^{\pi/2} = \frac{\sqrt{5}\,\pi^2}{8}$$

Q5 $\underline{V}(t) = \langle 1, 3t^2, 2t^3 \rangle$

$\underline{r}(t) = \langle t, t^3, \frac{1}{2}t^4 \rangle + \underline{c}(t)$

$\underline{r}(0) = \langle 0,0,0 \rangle + \underline{c}(t) = \langle 7,5,0 \rangle$

so $\underline{v}(t) = \langle t+7, t^3+5, \frac{1}{2}t^4 \rangle$

$\underline{v}(2) = \langle 9, 13, 8 \rangle$.

Q6 $P = IV \qquad V = IR$ so $P = \underset{①}{IV} = \underset{②}{I^2R} = \underset{③}{V^2/R}$

$V = 110,\ P = 200 \Rightarrow I = \cancel{110/200}\ 200/110 \simeq \cancel{0.55}\ 1.82 \qquad R = \cancel{110/0.55}\ 110/1.82 = \cancel{60.5}\ 60.5$

① $\Delta P \simeq V\Delta I + I\Delta V \simeq 110(\cancel{0.055}\ 0.182) + (\cancel{0.55}\ 1.82)\, 11 \simeq 40$

② $\Delta P \simeq 2IR\Delta I + I^2\Delta R \simeq 2(\cancel{0.55}\ 1.82)60.5(0.182) + (1.82)^2 6.05 \simeq 60$

③ $\Delta P \simeq \frac{2V}{R}\Delta V + -\frac{V^2}{R^2}\Delta R \simeq \frac{220}{60.5} 11 - \frac{(110)^2}{(60.5)^2} 6.05 \simeq 20$

measure V and R.

Q7 $f(x,y,z) = e^{2x-y} + \tan(yz)$

a) $\nabla f = \langle 2e^{2x-y}, -e^{2x-y} + z\sec^2(yz), y\sec^2(yz) \rangle$

$\nabla f(1,2,0) = \langle 2, -1, 2 \rangle$

b) $2(x-1) - (y-2) + 2(z-0) = 0.$

Q8 $f(x,y) = 3xy - x^2y - xy^2$

$$\left.\begin{aligned} f_x &= 3y - 2xy - y^2 = 0 \\ f_y &= 3x - x^2 - 2xy = 0 \end{aligned}\right\} \quad \begin{aligned} y(3-2x-y) &= 0 \\ x(3-x-2y) &= 0 \end{aligned}$$

$\underline{x=0}$: $y(3-y)=0$ $\quad (0,0) \quad (0,3)$

$\underline{y=0}$: $x(3-x)=0$ $\quad (0,0) \quad (3,0)$

$\underline{x\neq 0, y\neq 0}$: $\left.\begin{aligned} &① \ 3-2x-y=0 \\ &② \ 3-x-2y=0 \end{aligned}\right\}$ $\quad ①-2②: \ -3+3y=0 \quad \begin{aligned} y&=1 \\ x&=1 \end{aligned} \quad (1,1)$

$f_{xx} = -2y$

$f_{xy} = 3-2x-2y$

$f_{yy} = -2x$

$D = (f_{xx})(f_{yy}) - (f_{xy})^2$

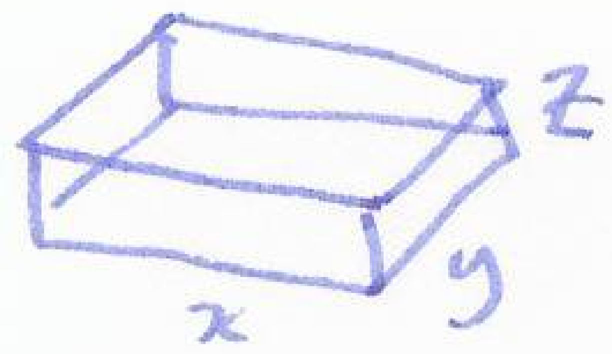
$D(0,0) = 0.0 - 9 < 0$ saddle.

$D(0,3) = -6.0 - (-3)^2 < 0$ saddle.

$D(3,0) = 0.-6 - (-3)^2 < 0$ saddle.

$$\left.\begin{aligned} D(1,1) &= (-2)(-2) - (-1)^2 > 0 \\ f_{xx} &< 0 \end{aligned}\right\} \text{ maximum}$$

Q9



$V = xyz = 4000$

$H = xy + 2yz + 2xz$ ← minimize

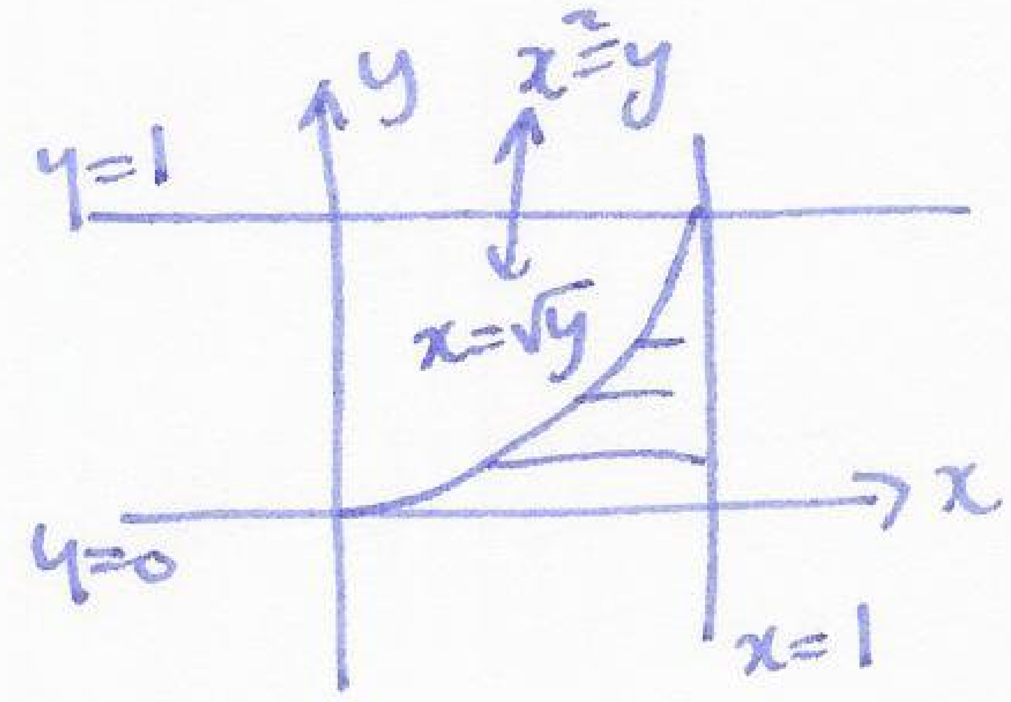
$\nabla H = \lambda \nabla V$: $\quad \nabla H = \langle y+2z,\ x+2z,\ 2x+2y \rangle$

$xyz = 4000$ $\quad \nabla V = \langle yz,\ xz,\ xy \rangle.$

$$\left.\begin{aligned} y + 2z &= \lambda yz \\ x + 2z &= \lambda xz \\ 2x + 2y &= \lambda xy \end{aligned}\right\} \quad \begin{aligned} xy + 2xz &= \lambda xyz = \lambda 4000 \\ xy + 2yz &= \lambda xyz = \lambda V \\ 2xz + 2yz &= \lambda xyz = \lambda V \end{aligned} \quad \begin{aligned} 2xz &= 2yz \Rightarrow x = y \\ xy &= 2xz \Rightarrow y = 2z \end{aligned}$$

$2x^3 = 4000$ so $x = \sqrt[3]{2000} = y \quad z = \frac{1}{2}\sqrt[3]{2000}.$

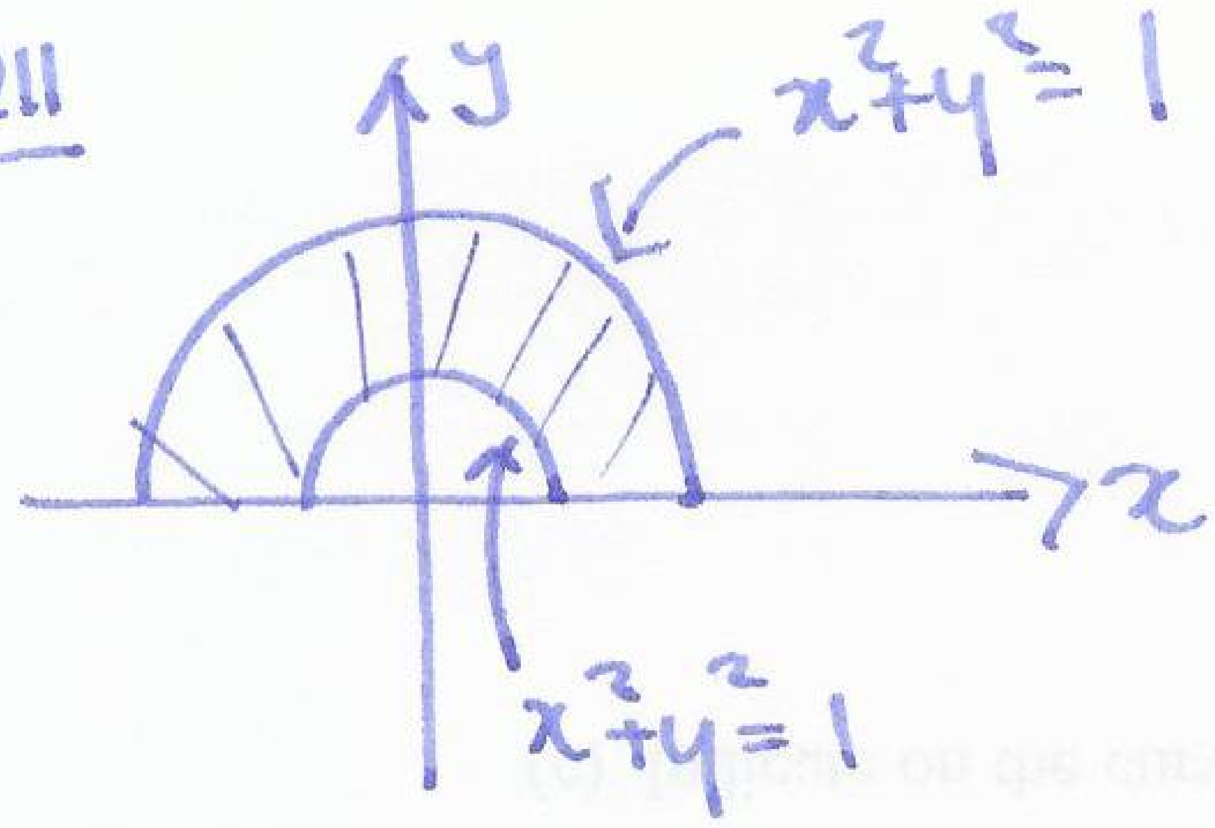
Q10 $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{y e^{x^2}}{x^3}\,dx\,dy$



$$\int_0^1 \int_0^{x^2} \frac{y e^{x^2}}{x^3}\,dy\,dx = \int_0^1 \left[\frac{1}{2} y^2 \frac{e^{x^2}}{x^3}\right]_0^{x^2} dx$$

$$= \int_0^1 \frac{1}{2} x e^{x^2}\,dx = \left[\frac{1}{4} e^{x^2}\right]_0^1 = \frac{1}{4}(e-1).$$

Q11



$\rho(x,y) = x^2+y^2$

use polars $\quad x = r\cos\theta$

$\quad y = \sin\theta$

$$\text{mass} = \iint_D \rho\, dA = \int_0^{\pi}\int_1^2 r^2\, r\, dr\, d\theta = 2\pi\left[\tfrac{1}{4}r^4\right]_1^2 = 2\pi\left(4-\tfrac{1}{4}\right) = \frac{30\pi}{4} = 15\pi/4.$$

$$M_x = \iint_D y\rho\, dA = \int_0^{\pi}\int_1^2 r^4 \sin\theta\, dr\, d\theta$$

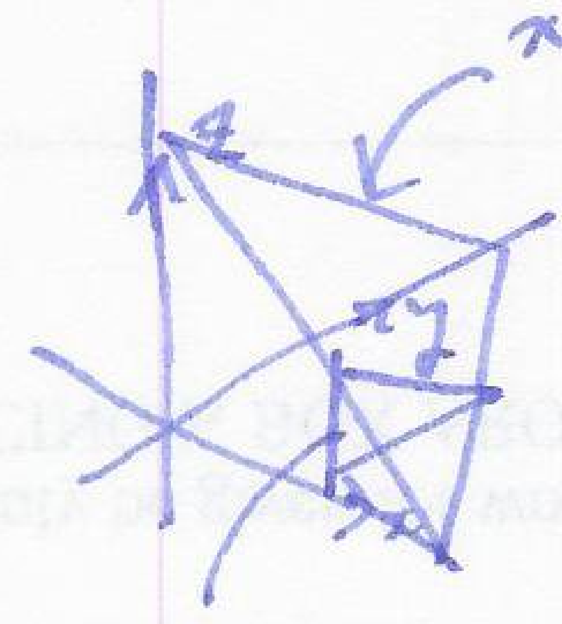
$$\int_0^{\pi} \sin\theta\, d\theta = \left[-\cos\theta\right]_0^{\pi} = -(-1)-(-1) = 2$$

$$\int_1^2 r^4\, dr = \left[\tfrac{1}{5}r^5\right]_1^2 = \tfrac{1}{2}(32-1) = \tfrac{31}{2}.$$

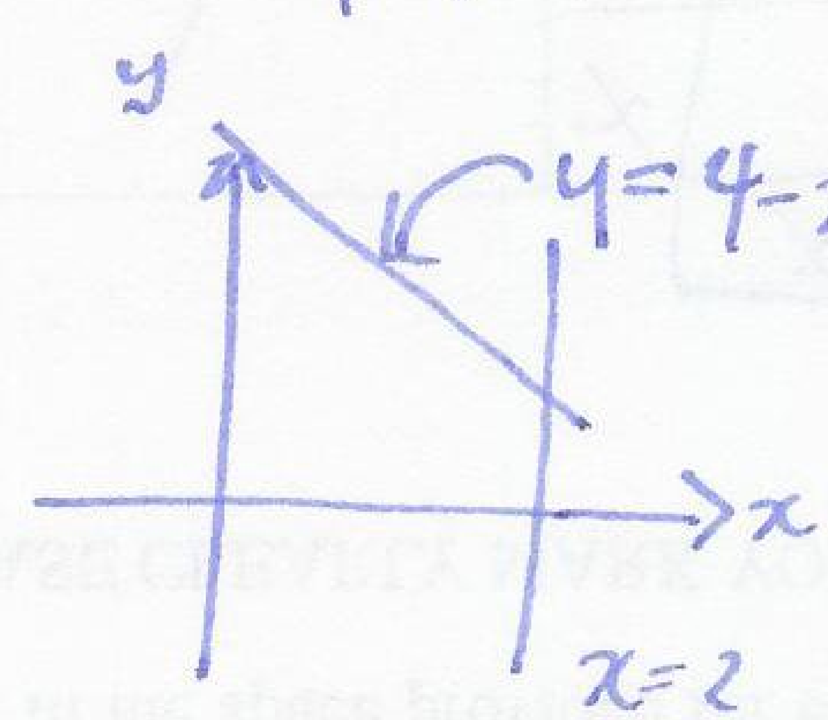
$$(\bar{x}, \bar{y}) = \left(0, \frac{62}{15\pi}\right).$$

Q12

$$\iiint_E dV$$



projection into xy-plane



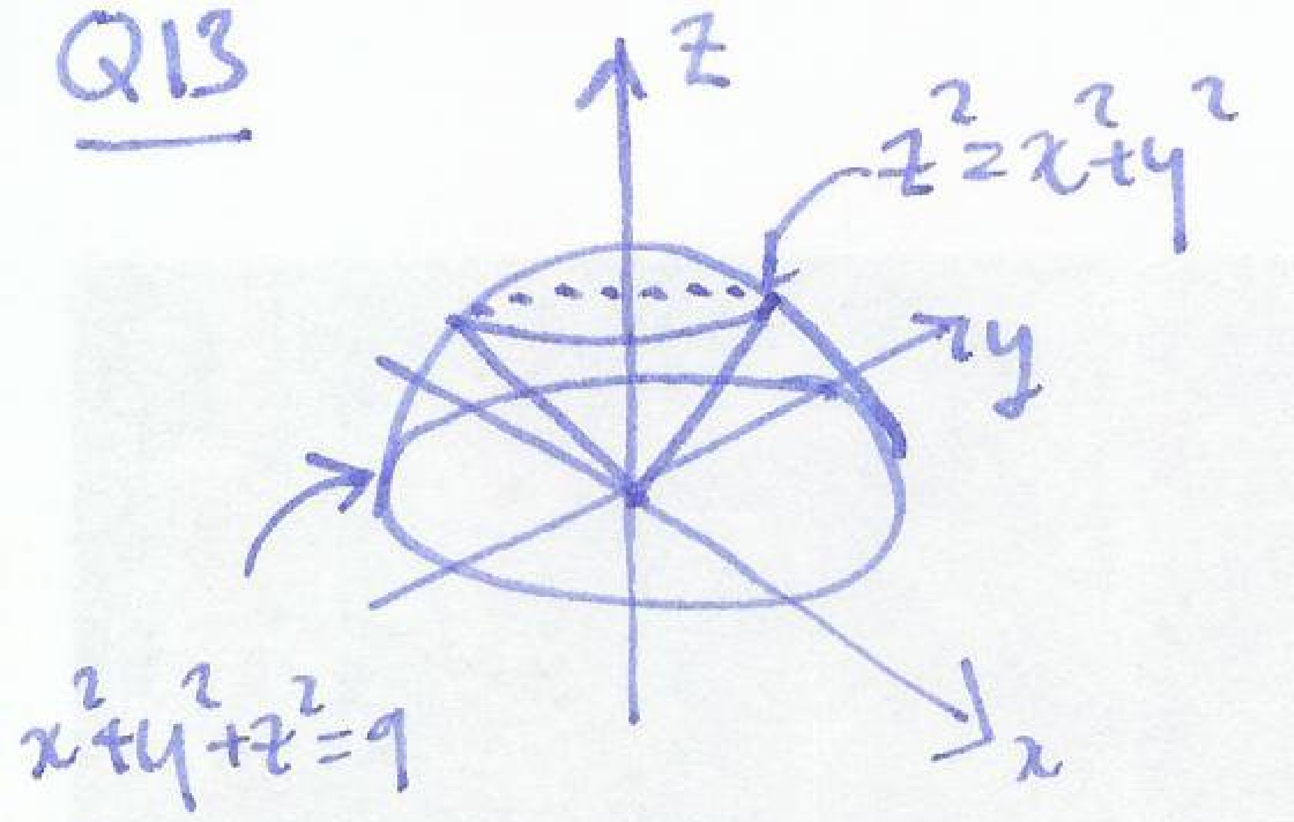
$$\int_0^2 \int_0^{4-x} \int_0^{4-x-y} dz\, dy\, dx \qquad \left[z\right]_0^{4-x-y} = 4-x-y$$

$$\int_0^{4-x} 4-x-y\, dy$$

$$\left[4y - xy - \frac{1}{2}y^2\right]_0^{4-x} = (4-x)^2 - \frac{1}{2}(4-x)^2 = \frac{1}{2}(4-x)^2$$

$$\int_0^2 \frac{1}{2}(4-x)^2\,dx = \left[-\frac{1}{6}(4-x)^3\right]_0^2 = 0 + \frac{1}{6}4^3 = \frac{32}{3}.$$

Q13



$\updownarrow$ $\rho = 3$

use spherical coordinates

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$z^2 = x^2 + y^2$

$\rho^2\cos^2\phi = \rho^2 \sin^2\phi.$

$\tan\phi = 1.$

$\phi = \pi/4.$

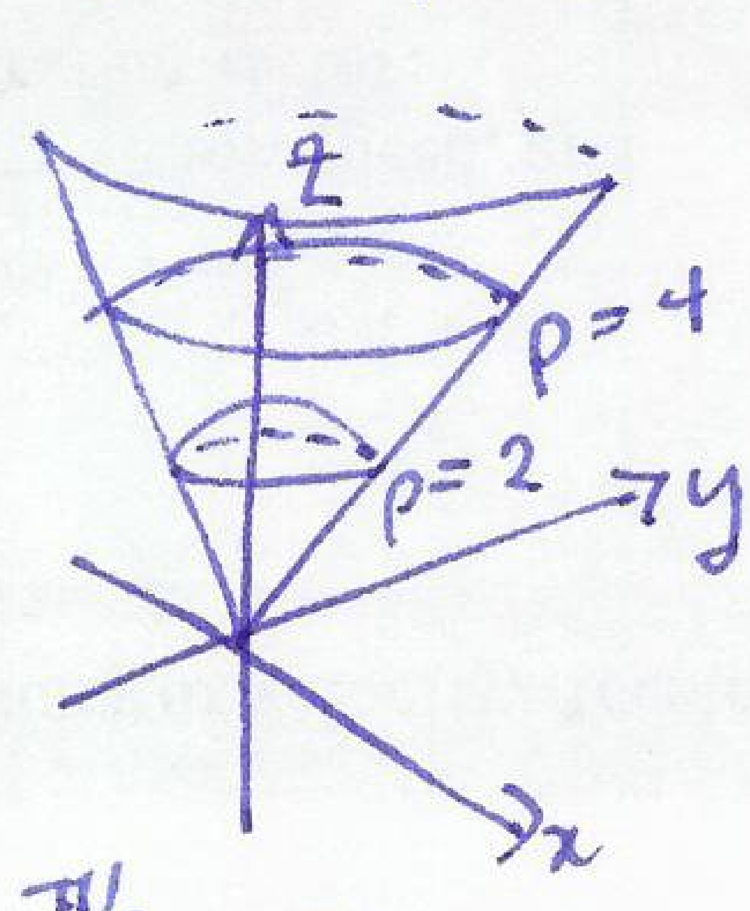
$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^3 \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$\int_0^3 \rho^2\, d\rho = \left[\frac{1}{3}\rho^3\right]_0^3 = 9$$

$$\int_0^{2\pi} 9\sin\phi\, d\theta = 18\pi\sin\phi$$

$$\int_0^{\pi/4} 18\pi \sin\phi\, d\phi = 18\pi\left[-\cos\phi\right]_0^{\pi/4} = 18\pi\left(1 - \sqrt{2}/2\right).$$

Q14



$x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\phi$

$$\int_0^{\pi/3}\int_0^{2\pi}\int_2^4 \rho^3 \cos\theta\sin\theta\sin^2\phi\cos\phi\;\rho^2\sin\phi\;d\rho\,d\theta\,d\phi$$

$$\int_2^4 \rho^5 \, d\rho = \left[\frac{1}{6}\rho^6\right]_2^4 = \frac{1}{6}\left(4^6-2^6\right)$$

$$\int_0^{2\pi} \cos\theta\sin\theta \, d\theta = \int_0^{2\pi} \frac{1}{2}\sin 2\theta \, d\theta = \left[-\frac{1}{4}\cos 2\theta\right]_0^{2\pi} = 0.$$

$$\int_0^{\pi/3} \sin^3\phi\cos\phi \, d\phi = \left[\frac{1}{4}\sin^4\phi\right]_0^{\pi/3} = \frac{1}{4}\left(\frac{9}{16}-0\right) = \frac{9}{64}$$

final answer: $\frac{1}{6}\left(4^6-2^6\right)\cdot 0 \cdot \frac{9}{64} = 0.$