Random Heegaard splittings

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Every 3-manifold has a Heegaard splitting

The gluing map is an element of the mapping class group $G = \text{MCG}(\Sigma) = \text{Diff}^+(\Sigma)/\text{Diff}_0(\Sigma)$

$G$ is finitely generated, let $\Gamma$ be a Cayley graph for $G$, consider the nearest neighbour random walk on $\Gamma$.

$$F_2 = \langle a, b \mid \rangle$$
More generally: pick a probability distribution \( \mu \) on \( G \), and define a Markov chain with transition probabilities \( p(x, y) = \mu(x^{-1}y) \).

Path space: \((\Omega, \mathbb{P})\), where \( \Omega = G^\mathbb{N} \), \( \mathbb{P} \) depends on \( \mu \)

Require:
- subgroup generated by \( \text{supp}(\mu) \) non-elementary
- \( \mu \) has finite first moment

Conjecture [Dunfield-W. Thurston]: Let \( w_n \) be a random walk of length \( n \) on the mapping class group. Then
- \( \mathbb{P}(M(w_n) \text{ is hyperbolic}) \to 1 \) as \( n \to \infty \)
- \( \text{Vol}(M(w_n)) \) grows linearly in \( n \)
[Kesten, Day et al.] A random walk on a non-amenable group has a linear rate of escape.

The mapping class group is non-amenable, so $|w_n|_\Gamma$ grows linearly, i.e. $\mathbb{P}(n/E \leq |w_n|_\Gamma \leq nE) \to 1$, as $n \to \infty$.

Complex of curves $\mathcal{C}(\Sigma)$:
- vertices: isotopy classes of simple closed curves
- simplices: spanned by disjoint collections of simple closed curves

Thm [M]: A random walk on the mapping class group makes linear progress in the complex of curves, i.e.
$\mathbb{P}(n/L \leq d_{\mathcal{C}}(x_0, w_nx_0) \leq Ln) \to 1$ as $n \to \infty$
[Brock-Souto] Hyperbolic volume coarsely equivalent to distance between disc sets in a (modified) pants complex

[Masur-Minsky] $\mathcal{C}(\Sigma)$ is $\delta$-hyperbolic.

[Klarreich] $\partial \mathcal{C}(\Sigma) = \text{minimal foliations} \subset \mathcal{PMF}$

[Kaimanovich-Masur] $w_n x_0 \to F \in \mathcal{PMF}$ a.s.

This gives a measure $\nu$ on $\mathcal{PMF}$, $\nu(X) = \text{probability } w_n \text{ converges to } F \in X$

Half space $H(x_0, y) = \{ z \in \mathcal{C}(\Sigma) \mid d_{\mathcal{C}}(z, y) \leq d_{\mathcal{C}}(z, x_0) \}$
Lemma: \( \nu(H(x_0, y)) \leq L|y| \), for some constant \( L < 1 \) independent of \( y \)

Proof: \( \nu(H(x_0, y)) \leq 1 - \epsilon \) for all \( |y| \geq K \), for constants \( \epsilon > 0 \) and \( K \)

Nested half spaces, conditional probability.
Linear progress:

For large enough \( m \), \(-mL^m + (1 - L) \geq a > 0\)

So \( \mathbb{E}(w_{n+m}) \geq \mathbb{E}(w_n) + a \)

use: Kingman’s subadditive ergodic theorem