Random walks on groups

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Aug 2013

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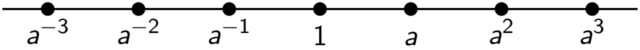
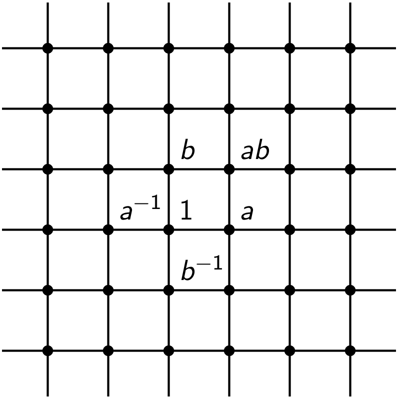
Basic example: Group $G = \mathbb{Z} = \langle a \mid \rangle$.

Cayley graph:
- vertices: group elements
- edges: $g \leftrightarrow h$ if $g^{-1}h$ is a generator



Nearest neighbour random walk on the Cayley graph.

$G = \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$



Let μ be a probability distribution on G.

Let s_i be independent μ-distributed random variables.

$$w_n = s_1 s_2 \dots s_n$$

Step space: $(G, \mu)^{\mathbb{N}}$

$$\phi\colon (s_i)_{i \in \mathbb{N}} \mapsto (w_i)_{i \in \mathbb{N}}$$

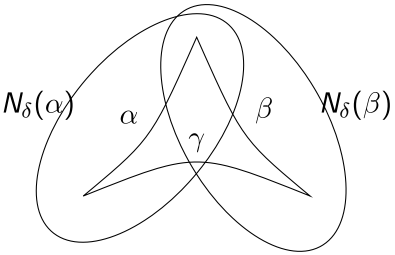
Path space: $(G^{\mathbb{N}}, \mathbb{P})$

$$\mathbb{P} = \phi_* \mu^{\mathbb{N}}$$

Poisson boundary $\leftrightarrow$ ergodic components of shift map.

[supp(μ) finite, generates non-elementary subgroup]

G acts by isometries on X Gromov hyperbolic: geodesic metric space with *thin triangles*:

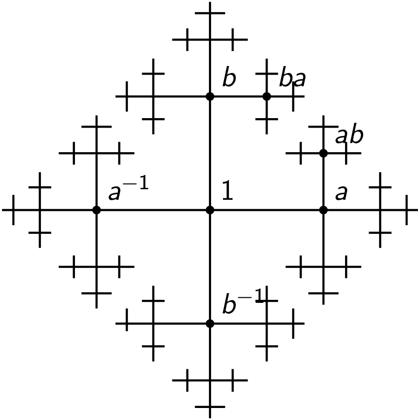


For any geodesic triangle any side is contained in a δ-neighbourhood of the other two.

Action non-elementary: two independent hyperbolic isometries.

X not necessarily locally compact.

$F_2 = \langle a, b \mid \rangle$



Boundary: space of ends of the tree $\leftrightarrow$ geodesic rays based at 1.

(w_n) converges to the boundary a.s. $\rightsquigarrow$ hitting measure ν on ∂X.

G Gromov hyperbolic:

• (w_n) converges to the boundary a.s. and $(\partial X, \nu)$ is the Poisson boundary [Kaimanovich]

• Linear progress [Guivarc'h]

$$\lim_{n\to\infty} \frac{d(1, w_n)}{n} = L > 0, \text{ a.s.}$$

• translation length $\tau(w_n)$ grows linearly in n

$$\tau(g) = \lim_{n\to\infty} \frac{d(x, g^n x)}{n}$$

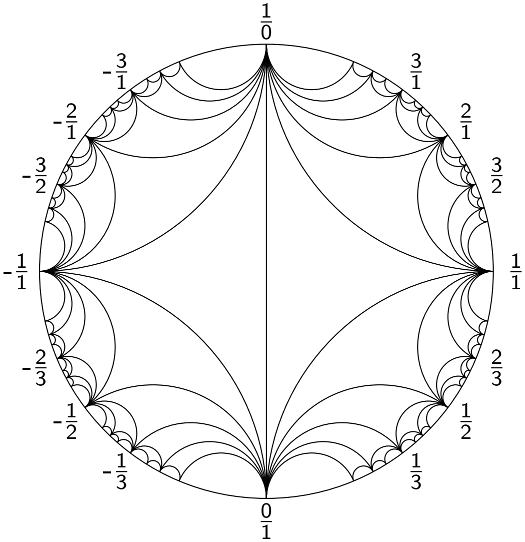
• stable commutator length grows as $n/\log n$ [Calegari-M] [Calegari-Walker]

Examples:

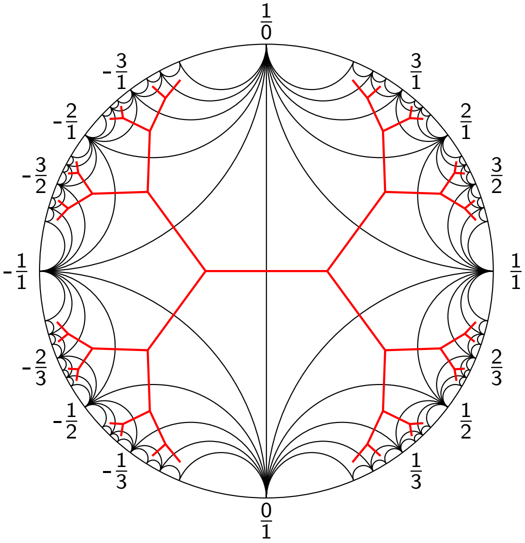
- Free groups $F_n = \langle a_1, \ldots a_n \mid \rangle$
- Fundamental groups of compact negatively curved manifolds
- Gromov hyperbolic groups
- Fundamental groups of finite volume negatively curved manifolds
- Relatively hyperbolic groups
- Mapping class groups of surfaces
- $\text{Out}(F_n)$

Other boundaries:
[Karlsson-Ledrappier] [Gouezel-Lalley] [Gouezel]

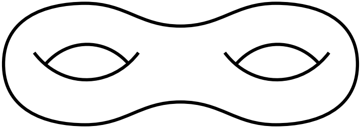
Example: PSL(2,$\mathbb{Z}$)



Example: PSL(2,$\mathbb{Z}$)

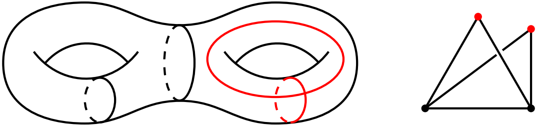


Mapping class group of a surface S



X complex of curves $\mathcal{C}(S)$

- vertices: isotopy classes of simple closed curves
- simplices: disjoint collections of simple closed curves



$\mathcal{C}(S)$ Gromov hyperbolic [Masur-Minsky], not locally finite

G mapping class group of surface

- Convergence to the boundary [Kaimanovich-Masur] [Klarreich]
- Linear progress in $\mathcal{C}(S)$ [M]

Generic elements are pseudo-Anosov [Rivin] [Kowalski] [M]

$$0 \to \mathcal{T} \to G \to Sp(2g, \mathbb{Z}) \to 0$$

Torelli [Malestein-Souto] [Lubotzky-Meiri] [M]

- translation length on $\mathcal{C}(S)$ grows linearly [M]
- stable commutator length (scl) grows as $n/\log n$ [Calegari-M]

Out(F_n): Aut(F_n) /conjugacy

X complex of free factors [Bestvina-Feighn]

[Calegari-M]:

- Convergence to the boundary
- Linear progress
- translation length grows linearly [Sisto]
- scl grows as n / log n [Osin]

Horofunction compactification: $\rho\colon X \hookrightarrow C(X, \mathbb{R})$ (compact-open topology)

$$x \mapsto d(x,y) - d(x,x_0)$$

$$\overline{X}^h = \overline{\rho(X)}$$

Compact, weak limit ν of $\mu^{(n)}$ exists

There is a "local minimum" map $\phi\colon \overline{X}^h \to X \cup \partial X$.

Show $\nu(\phi^{-1}(X)) = 0$.

use [Kaimanovich] ([Margulis] [Furstenberg])