Random walks on graphs and groups

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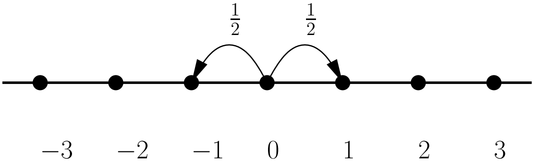
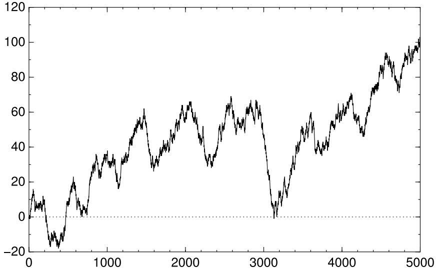
A random walk on $\mathbb{Z}$



At time $t = 0$ start at $w_0 = 0$

$$w_{t+1} = \begin{cases} w_t + 1 \text{ with probability } 1/2 \\ w_t - 1 \text{ with probability } 1/2 \end{cases}$$



	-4	-3	-2	-1	0	1	2	3	4	
$t=0$					1					
$t=1$				1	0	1				/2
$t=2$			1	0	2	0	1			/4
$t=3$		1	0	3	0	3	0	1		/8
$t=4$	1	0	4	0	6	0	4	0	1	/16

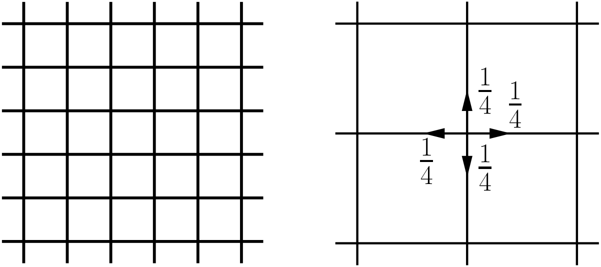
In general $\mathbb{P}(w_t = t - 2k) = \frac{1}{2^t} \binom{t}{k}$

Average distance from 0 is $\mathbb{E}(|w_t|) \sim \sqrt{t}$

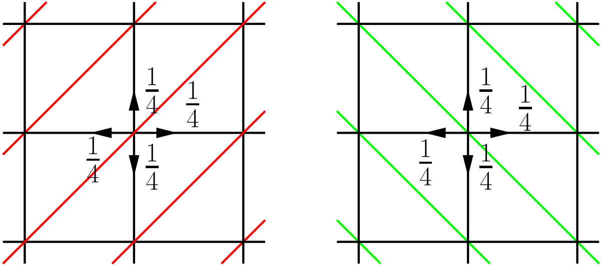
$\mathbb{P}(w_t = 0) \sim \frac{1}{\sqrt{t}} \implies \mathbb{P}(w_t \text{ hits 0 infinitely often}) = 1$

We say the random walk on $\mathbb{Z}$ is *recurrent*.

A random walk on $\mathbb{Z}^2$

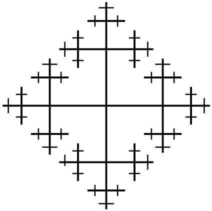


This is really two independent random walks on $\mathbb{Z}$, so $\mathbb{P}(w_t = (0,0)) \sim \frac{1}{t}$.



The nearest neighbour random walk on a (finite valence) graph:

- Start at a particular vertex v_0 at time 0.
- At time t jump to one of your nearest neighbours, chosen with equal probability.



The random walk on a four-valent tree is *transient*, i.e.

$$\mathbb{P}(\text{random walk hits } v_0 \text{ finitely often}) = 1.$$

The random walk makes linear progress, $\mathbb{E}(d(v_0, w_t)) \sim t$.

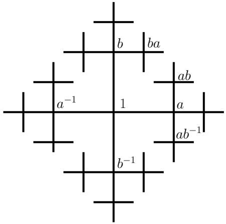
Random walks on groups:

Pick a (symmetric) generating set A.

The *Cayley graph* of a finitely generated group is the graph with

- vertices: elements of the group
- edges: connect elements which differ by a generator

The graph depends on the choice of generating set A, but any two choices give quasi-isometric graphs.



$F_2 = \langle a, b \mid \ \rangle$

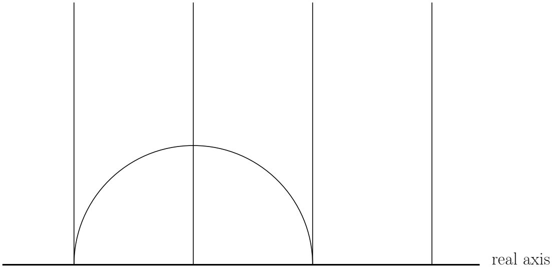
group elements: aba^{-1}

$(aba^{-1})(ab) = abaa^{-1}b = ab^2$

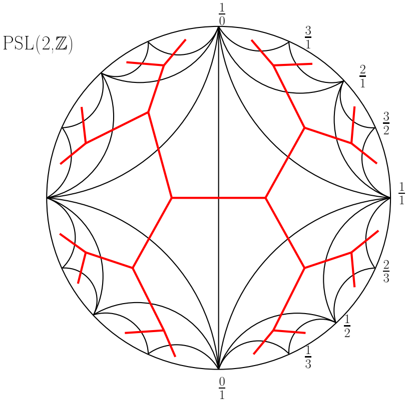
Thm[Kesten, Day]: A random walk on a group has a linear rate of escape iff the group is non-amenable

$SL(2,\mathbb{Z})$: 2×2 integer matrices with determinant $+1$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ acts on $\mathbb{C}$ by $z \mapsto \frac{az+b}{cz+d}$, preserves upper half space.



$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \leftrightarrow z \mapsto z+1$ $\qquad$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \leftrightarrow z \mapsto -1/z$

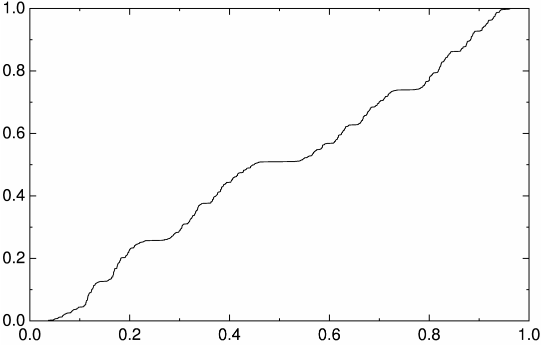


Sample paths converge to the boundary with probability one.

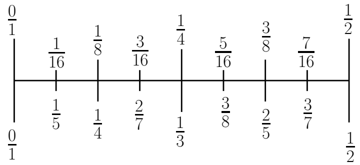
This gives a measure on the boundary, called *harmonic measure* ν.

$\nu(X) = \mathbb{P}$(probability you converge to X)

Harmonic measure is *not* Lebesgue measure



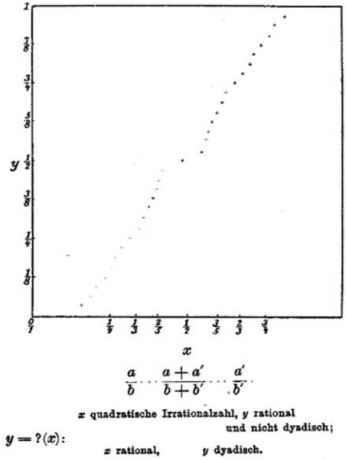
$$? : \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \mapsto \overbrace{0.0\ldots0}^{a_1}\overbrace{1\ldots1}^{a_2}\ldots$$



Lebesgue measure: $\mathbb{P}(a_i = n) \sim \frac{1}{n^2}$

Harmonic measure: $\mathbb{P}(a_i = n) \sim \frac{1}{2^n}$

Fig. 7.
Kriterium für die reellen quadratischen Irrationalzahlen.



Hermann Minkowski, 1904.

Generic elements in groups.

A subset $X \subset G$ is *generic* if it has

- High probability:

$$\mathbb{P}(w_n \in X) \to 1, \text{ as } n \to \infty.$$

- High density:

$$\frac{|X \cap B_n(1)|}{|G \cap B_n(1)|} \to 1, \text{ as } n \to \infty.$$

- High density with respect to some other metric on G.

Example: $F_2 \times 0 \subset F_2 \times \mathbb{Z}$

Convergence to the boundary works for:
matrix groups, e.g. SL(n,$\mathbb{Z}$) [Furstenberg]

- random matrices are irreducible [Rivin][Kowalski]

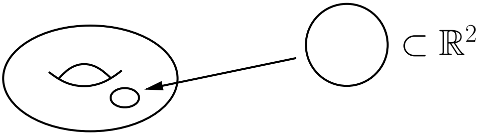
δ-hyperbolic groups [Kaimanovich-Woess]

- random elements are hyperbolic,
  translation length tends to infinity

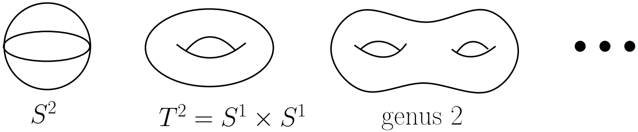
Mapping class groups, braid groups [Kaimanovich-Masur]

- random elements are pseudo-Anosov [Rivin][Kowalski][M]

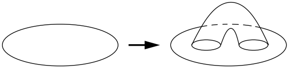
Surface or 2-manifold: space locally modelled on $\mathbb{R}^2$



Classification of surfaces

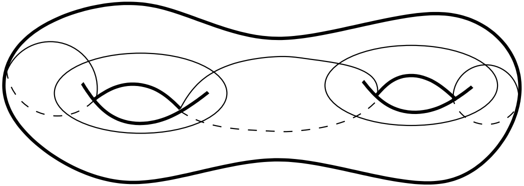


Add handles:

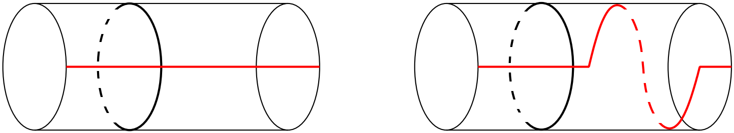


The mapping class group of a surface Σ is

$$G = \{\text{surface homeomorphisms}\}/\text{isotopy}.$$

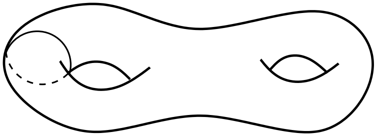


The mapping class group is finitely generated by Dehn twists.

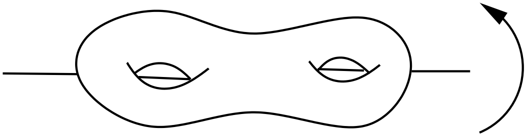


Thurston's classification of surface homeomorphisms

Reducible:

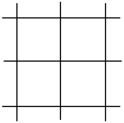


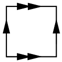
Periodic:



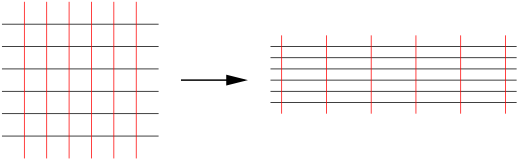
Pseudo-Anosov: everything else

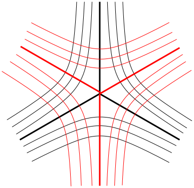
Anosov: $A \in SL(2,\mathbb{Z})$ with trace > 2, e.g. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.



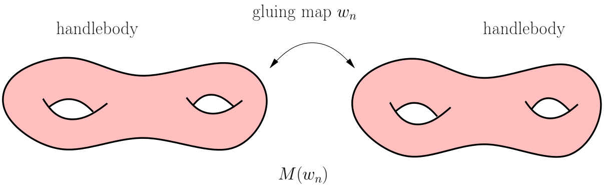


Pseudo-Anosov: e.g. branched cover of an Anosov map.





Application to 3-manifolds: Heegaard splittings



- [M] $\mathbb{P}(M(w_n)$ is hyperbolic$) \to 1$ as $n \to \infty$.
- [M] $\mathrm{vol}(M(w_n))$ grows linearly n.

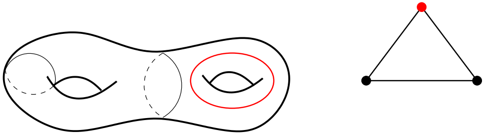
[Dunfield-W. Thurston] $\mathbb{P}(M(w_n)$ is $\mathbb{Q}-$ homology sphere$) \to 1$.

[Dunfield-D. Thurston] $\mathbb{P}(M(w_n)$ is fibered$) \to 0$. (genus 2)

The mapping class group G acts on the complex of curves $\mathcal{C}(\Sigma)$.

$\mathcal{C}(\Sigma)$ is a simplicial complex.

- vertices: isotopy classes of simple closed curves.
- simplices: spanned by disjoint simple closed curves.



Finite dimensional, but not locally finite.

[Masur-Minsky] $\mathcal{C}(\Sigma)$ is δ-hyperbolic.

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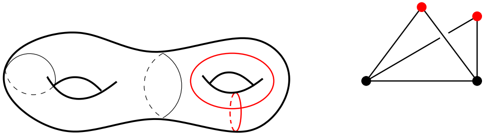
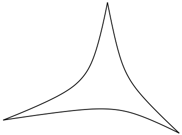


Finite dimensional, but not locally finite.

[Masur-Minsky] $\mathcal{C}(\Sigma)$ is δ-hyperbolic.

[Gromov] A metric space is δ-hyperbolic if every geodesic triangle is δ-thin, i.e. any side is contained in a δ-neighbourhood of the other two.



Examples: hyperbolic space, trees, the complex of curves $\mathbb{C}(S)$.

Isometries of δ-hyperbolic spaces are:

- elliptic, fix a point in the interior (periodic, reducible)
- parabolic (none of these in G)
- hyperbolic (pseudo-Anosov)