Random Heegaard splittings

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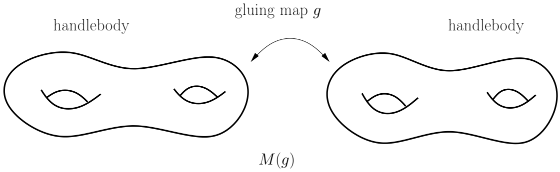
Heegaard splitting: decomposition of a closed 3-manifold into two handlebodies.

Handlebody: regular neighbourhood of a graph in $\mathbb{R}^3$.

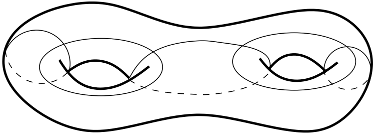


Every closed 3-manifold has a Heegaard splitting.

Isotopic gluing maps give homeomorphic 3-manifolds.

The gluing map is an element of the mapping class group of the surface S:

$G = \mathrm{MCG}(S) = \{$ surface homeomorphisms $\}/$ isotopy.



The mapping class group is finitely generated.

Random Heegaard splitting: $M(w_n)$, where w_n is a random word of length n in the mapping class group G.

Random word: Fix generating set $A = \{a_1, \ldots a_k\}$.
Choose s_i from A with uniform probability.

$$w_n = s_1 s_2 s_3 \ldots s_{n-1} s_n$$

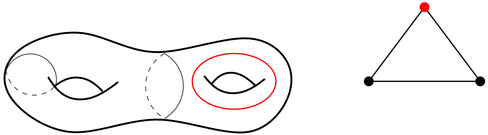
Equivalently: w_n is the nearest neighbour random walk of length n on the Cayley graph for G with respect to the generating set A.

Thm[M]: $\mathbb{P}(M(w_n)$ is hyperbolic$) \to 1$ as $n \to \infty$.

The mapping class group acts on the complex of curves $\mathcal{C}(S)$.

The complex of curves is a simplicial complex.

- vertices: isotopy classes of essential simple closed curves.
- simplices: spanned by disjoint simple closed curves.



Finite dimensional, but not locally finite.

G acts by simplicial isometries on $\mathcal{C}(S)$.

Disc set: collection of simple closed curves in the boundary surface which bound discs in the handlebody.

A Heegaard splitting $M(g)$ has two handlebodies, with disc sets Δ and $g\Delta$.

Splitting distance: minimum distance in $\mathcal{C}(S)$ between Δ and $g\Delta$.

[T. Kobayashi;Hempel] If the splitting distance is more than two, then M is irreducible, atoroidal and not Seifert fibered.

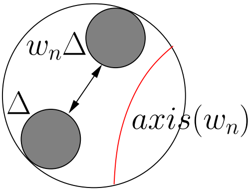
[Perelman] Geometrization $\Rightarrow$ M is hyperbolic, if splitting distance > 2.

[Masur-Minsky] the complex of curves is δ-hyperbolic.

hyperbolic isometries of $\mathcal{C}(S) \leftrightarrow$ pseudo-Anosov elements

[Rivin]: The probability that w_n is pseudo-Anosov tends to 1 as $n \to \infty$.

[Masur-Minsky]: The disc set is quasiconvex.



[M]: Translation length of w_n on $\mathcal{C}(S)$ grows linearly in n.

Gromov boundary: $\{$ set of quasi-geodesic rays $\}/\sim$
Two rays are equivalent if they stay a bounded distance apart.

Pick basepoint x_0 in $\mathcal{C}(S)$.

[Kaimanovich-Masur, Klarreich] $\{w_n x_0\}$ converges to the Gromov boundary with probability one.

This gives a hitting measure or harmonic measure ν on Gromov boundary.

[Kerckhoff] Limit set of Δ has harmonic measure zero.

Need to understand joint distribution of endpoints of axes.

For g pseudo-Anosov, let

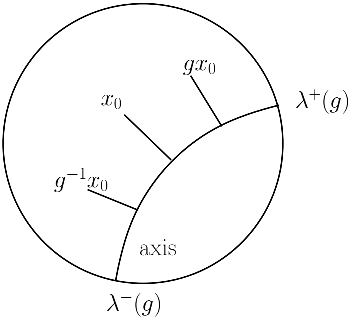
$$(\lambda^+(g), \lambda^-(g)) \in \partial\mathcal{C}(S) \times \partial\mathcal{C}(S)$$

be the endpoints of the axis of g.

Let L_n be the distribution of endpoints of axes of pseudo-Anosov random words of length n.

Claim: $L_n \to \nu \times \nu$ as $n \to \infty$.

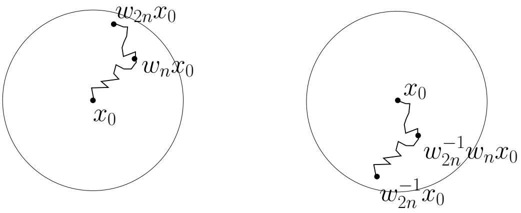
If the translation length of g is bigger than $K(\delta)$, then $\lambda^+(g)$ is close to gx_0, and $\lambda^-(g)$ is close to $g^{-1}x_0$, in the visual metric based at x_0.



So $L_n \sim (w_n, w_n^{-1})$.

In visual metric based at x_0:

$$w_n x_0 \text{ and } w_{2n} x_0 \text{ are close.}$$
$$w_{2n}^{-1} x_0 \text{ and } w_{2n}^{-1} w_n x_0 \text{ are close.}$$



So $(w_{2n}, w_{2n}^{-1}) \sim (w_n, w_{2n}^{-1} w_n)$.

As $w_{2n} = s_1 \dots s_n s_{n+1} \dots s_{2n}$, then

$$w_n = s_1 \dots s_n \quad \text{and} \quad w_{2n}^{-1} w_n = s_{2n}^{-1} \dots s_{n+1}^{-1}$$

are independent.