Asymptotics for pseudo-Anosovs in Teichmüller lattices

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$S$ closed orientable surface

Def: $G = \text{MCG}(S) = \text{Homeo}^+(S)/\text{isotopy}$

[Thurston] Classification of elements of $G$:
  • Periodic
  • Reducible
  • Pseudo-Anosov
Teichmüller space $\mathcal{T}(S) \cong \mathbb{R}^{6g-6}$

- Space of conformal structures on $S$
- Space of hyperbolic structures on $S$

Teichmüller metric: $d_T(x, y) = \inf \frac{1}{2} \log K$

- $(\mathcal{T}, d_T)$ infinite diameter, complete
- $G$ acts by isometries on $\mathcal{T}$, properly discontinuously
- unique geodesic connecting any pair of points
- moduli space $\mathcal{T}/G$ finite volume
Teichmüller lattice: $G_y$

$|G_y \cap B_r(x)| \sim C(x, y)e^{hr}$

cf [Margulis]
Def: $R =$ non-pseudo-Anosov elements of $G$.

Thm[M]:
\[
\frac{|Ry \cap B_r(x)|}{|Gy \cap B_r(x)|} \to 0 \text{ as } r \to \infty.
\]

$Q =$ unit area quadratic differentials $= \text{"unit tangent bundle of } T\text{"}$

$g_t : Q \to Q$ geodesic flow

$\pi : Q \to T$, $S(x) = \pi^{-1}(x) = \text{visual boundary}$
bisector: $U \subset S(x), V \subset S(y)$

$g \in B(U, V) \iff q_x(gy) \in U \text{ and } q_y(\gamma^{-1}x) \in V$
Thm[ABEM]:

$$|Gy \cap B_r(x), g \in B(U, V)| \sim \frac{1}{h} e^{hr} \Lambda^+(U)\Lambda^-(V)$$

$\Lambda^+, \Lambda^-$ measures on $S(x), S(y)$ respectively, defined in terms of the Masur-Veech measure $\mu$ on $Q$, which is $g_t$-invariant, with $\mu(Q/G) = 1$.

Note: distribution of leaving directions $q_x(gy)$ given by $\Lambda^+$, distribution of arriving directions $q_y(g^{-1}x)$ given by $\Lambda^-$, independent.
Consider $R =$ set of non-pseudo-Anosov elements. 
$R_k = \{ g \in R \mid d_T(gy, g'y) \leq k, \text{ some } g' \in R \setminus g \}$ “$k$-dense”
$R \setminus R_k$ “$k$-separated”

Thm[M]: $\overline{R_k}$ has measure zero in visual boundary
Equidistribution:
Thm[Veech]: The Teichmüller geodesic flow is mixing.

\[
\lim_{t \to \infty} \int_{Q/G} \alpha(g_t q) \beta(q) d\mu(q) = \int_{Q/G} \alpha(q) d\mu(q) \int_{Q/G} \beta(q) d\mu(q)
\]

Conditional mixing:

\[
\lim_{t \to \infty} \int_{S(x)} \alpha(g_t q) \beta(q) ds_x(q) = \int_{Q/G} \alpha(q) d\mu(q) \int_{S(x)} \beta(q) ds_x(q)
\]

Here \( \alpha, \beta \) continuous, compact support.
Go back distance $k/2$ along geodesic from $x$ to $gy$, look for lattice point distance at most $d < k/2$ away, get at least $|Gy \cap B_{k/2-d}(y)|$ lattice points in $B_k(gy) \cap B_r(x)$.

i.e. this estimate works for the proportion of lattice points in $\partial B_{k/2}(y)$ which lie in $N_d(Gx)$, mixing implies this is $\text{vol}(N_d(x))$ in $Q/G$, tends to 1 as $d \to \infty$. 