Random walks on the mapping class group

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May 2008
• Random walks

• Random walks on the mapping class group

  Theorem: A random walk on the mapping class group gives a pseudo-Anosov element with asymptotic probability one.

• Random Heegaard splittings

  Theorem: A random Heegaard splitting is hyperbolic with asymptotic probability one.
A random walk on $\mathbb{Z}$

At time $t = 0$ start at $w_0 = 0$

$$w_{t+1} = \begin{cases} 
  w_t + 1 & \text{with probability } 1/2 \\
  w_t - 1 & \text{with probability } 1/2
\end{cases}$$
The nearest neighbour random walk on a (finite valence) graph:

- Start at a particular vertex at time 0.
- At time $n$ jump to one of your nearest neighbours, chosen with equal probability.

Random walks on groups:

Pick a (symmetric) generating set $A$.

The *Cayley graph* of a finitely generated group is the graph with

- vertices: elements of the group
- edges: connect elements which differ by a generator

The graph depends on the choice of generating set $A$, but any two choices give quasi-isometric graphs.
Example of a Cayley graph:

\[ F_2 = \langle a, b \mid \rangle \]

Key example: the nearest neighbour random walk on a Cayley graph of the mapping class group.

- Start at the identity at time 0.
- At time \( n \) jump to one of your nearest neighbours, chosen with equal probability.
More generally: pick a probability distribution $\mu$ on $G$. Consider the Markov chain with set $G$, and transition probabilities $p(x, y) = \mu(x^{-1}y)$.

Time 0: start at identity.
Time 1: distributed according to $\mu$.
Time 2: distributed according to $\mu^2 = \text{convolution of } \mu \text{ with itself}$.

$$\mu^2(x) = \sum_{y \in G} \mu(y)\mu(y^{-1}x)$$

Time $n$: distributed according to $\mu^n$, $n$-fold convolution of $\mu$ with itself.
Path space: \((G^\mathbb{Z}^+, \mathbb{P})\), probability space.

\(G^\mathbb{Z}^+\) infinite product of \(G\)'s.

A sample path \(\omega \in G^\mathbb{Z}^+\) is an infinite sequence of group elements corresponding to the locations of the random walk.

Projection \(w_n : G^\mathbb{Z}^+ \to G\) to the \(n\)-th factor is a random variable which gives the location of the sample path at time \(n\).

The distribution of \(w_n\) is given by \(\mu^n\).

[Kolmogorov] This determines \(\mathbb{P}\).

Key point: this enables us to talk about infinite length random walks.
Example: \( \text{PSL}(2, \mathbb{Z}) \)

Sample paths converge to the boundary with probability one. This gives a measure on the boundary, called \textit{harmonic measure} \( \nu \).

\[ \nu(X) = \mathbb{P}(\text{sample paths which converge to points in } X) \]
This harmonic measure on $S^1$ is \textit{not} Lebesgue measure.

\[ \begin{align*}
\frac{1}{a_1 + \frac{1}{a_2 + \ldots}} &\mapsto 0.0\ldots01\ldots1\ldots \\
\end{align*} \]
Convergence to the boundary works for:

- matrix groups, e.g. \( \text{SL}(n, \mathbb{Z}) \) [Furstenberg]
  - random matrices are irreducible [Rivin, Kowalski]

- \( \delta \)-hyperbolic groups [Kaimanovich-Woess]
  - random elements are hyperbolic, translation length tends to infinity

- Mapping class groups, braid groups [Kaimanovich-Masur]
  - random elements are pseudo-Anosov [M]
The mapping class group of a surface $S$ is
\{surface diffeomorphisms\}/isotopy.
$G = \text{MCG}(S) = \text{Diff}^+(S)/\text{Diff}_0(S)$

The mapping class group is finitely generated by Dehn twists.
The surface $S$ may have boundary or punctures

The mapping class group of the $n$-punctured disc is also known as the braid group.

Thurston’s classification of surface homeomorphisms

- Reducible:

The map fixes a disjoint collection of simple closed curves.
• Periodic:

Some power of the map is isotopic to the identity.

• Pseudo-Anosov:

Everything else...
Useful facts about the mapping class group.

[Masur-Minsky] The mapping class group is weakly relative hyperbolic.

$G$ finitely generated by $A$, gives word metric on $G$ (same as Cayley graph metric).

$\hat{G} = G$ with word metric from an infinite generating set $A \cup \{H_i\}$.

In this case $H_i = \text{stab}(\alpha_i)$, where $\alpha_i$ are representatives of simple closed curves under the action of $G$.

If $\hat{G}$ is $\delta$-hyperbolic then we say that $G$ is weakly relatively hyperbolic (with respect to $\{H_i\}$).
Recall a metric space is $\delta$-hyperbolic if every geodesic triangle is $\delta$-thin, i.e. any side is contained in a $\delta$-neighbourhood of the other two.

Examples: hyperbolic space, trees, the complex of curves $\mathcal{C}(S)$.

[Masur-Minksy] show that the relative space $\hat{\mathcal{G}}$ is quasi-isometric to the complex of curves.
The complex of curves is a simplicial complex.

- vertices: isotopy classes of simple closed curves.
- simplices: spanned by disjoint simple closed curves.

Finite dimensional, but not locally finite.

[Masur-Minsky] the complex of curves is $\delta$-hyperbolic.
Isometries of $\delta$-hyperbolic spaces are
- elliptic, fix a point in the interior (periodic, reducible)
- parabolic (none of these)
- hyperbolic (pseudo-Anosov)

Gromov boundary: \{ set of quasi-geodesic rays \}/ ~
Two rays are equivalent if they stay a bounded distance apart.

[Klarreich] The Gromov boundary of the complex of curves is $F_{\text{min}}$
the space of minimal foliations in $PMF$, Thurston’s space of
projective measured foliations.

$PMF$ is a sphere of dimension $6g - 5$, $g =$ genus of $S$.

pseudo-Anosov maps act on $C(S) \cup F_{\text{min}}$ as translations along an
axis with a unique pair of fixed points, the attracting and repelling fixed points.
[Kaimanovich-Masur, + Klarreich] A random walk on the mapping class group converges almost surely to a uniquely ergodic foliation in $PMF$, as long as the support of $\mu$ is a non-elementary subgroup. The resulting harmonic measure $\nu$ on $F_{\text{min}}$ is non-atomic.

uniquely ergodic $\Rightarrow$ minimal

non-elementary: the subgroup contains a pair of pseudo-Anosov elements with distinct endpoints.

Recall $\nu(X) =$ proportion of sample paths which converge into $X$.

$\nu$ governs the long time behaviour of sample paths.
Theorem [Rivin, Kowalski]: The probability that \( w_n(\omega) \) is pseudo-Anosov tends to 1 as \( n \to \infty \).

Consider the action on homology, i.e. map from \( G \) to \( Sp(2g, \mathbb{Z}) \).

[Casson-Bleiler] If image of \( g \) is irreducible, no roots of unity as eigenvalues, characteristic polynomial not a power of a lower degree polynomial, then \( g \) is pseudo-Anosov.

Theorem [M]: The probability that the translation length of \( w_n(\omega) \) on \( C(S) \) is at most \( K \) tends to zero as \( n \to \infty \).

Requires support of \( \mu \) generates a non-elementary subgroup not contained in a centralizer.

Translation length of \( g \): \( \lim_{n} \frac{1}{n} d_{C(S)}(x, g^n x) \).
Sketch of proof.

Observation: if $X \subset G$ and limit set of $X$ has (harmonic) measure zero in $\mathcal{F}_{\min}$, then the random walk is transient on $X$. (A sample path hits $X$ finitely many times almost surely.)

Let $R =$ elements of $G$ of translation length at most $K$. Then $\nu(\overline{R}) = 1$.

Let $R_k =$ $k$-dense elements of $R$, i.e. $r \in R$ such that there is some other $r' \in R$ such that $d_G(r, r') \leq k$.

Claim: $\nu(\overline{R}_k) = 0$. 
\[ P(w_n(\omega) \in R) = P(w_n(\omega) \in R_k) + P(w_n \in R \setminus R_k) \]

- \( P(w_n(\omega) \in R_k) \to 0 \) as \( n \to \infty \) by transience.
- \( P(w_n(\omega) \in R \setminus R_k) \leq 1/k \)

True for all \( k \) implies \( P(w_n(\omega)) \to 0 \) as \( n \to \infty \).
More details:

\[ \overline{R}_k = \bigcup \overline{C(g)} \], where word length of \( g \) at most \( k \).

\( C(g) \) = centralizer of \( g \), i.e. \( h \in G \) such that \( gh = hg \).

- \( g \) pseudo-Anosov: \( C(g) \) virtually cyclic, limit set is fixed points.
- \( g \) reducible: centralizer bounded diameter in \( \hat{G} \), limit set empty.
- \( g \) periodic: \( C(g) \) lower dimensional sphere.

[ Nielsen] a finite cyclic subgroup of \( G \) fixes a point in Teichmüller space = set of hyperbolic structures on \( S \).

\( \Rightarrow \) finite cyclic groups realized by covering translations.

So fixed set is lower dimensional Teichmüller space inside original one, so limit set is a lower dimensional PMF inside original one.

[ distance reducing maps \( G \to \mathcal{T}(S) \to \hat{G} \) ]
Relative conjugacy bounds:

If $a$ and $b$ are conjugate in $G$ then there is a conjugating word $w$ such that $|\hat{w}| \leq K(|\hat{a}| + |\hat{b}|)$.


This implies if $g$ is conjugate to a short word $s$, and $w$ is a shortest conjugating word in the relative metric, then the path $ws\bar{w}^{-1}$ is a quasi-geodesic path, where the quasi-geodesic constants depend on the length of $s$. 
s has bounded length, so thin triangles implies if $w$ very long, then a final segment of $w$ fellow-travels with an initial segment of $w^{-1}$. So red path is a short conjugate of $s$, so could have chosen a shorter conjugating word.

If $r \in R_k$, then there is $g$ of word length at most $k$ such that $rg = r' \in R_k$, so $R_k$ is a finite union of $R \cap Rg$. 
Claim: \( \overline{R \cap Rg} = \overline{C(g)} \)

\[
r = wsw^{-1} \quad \text{and} \quad r' = w's'w'^{-1}, \quad \text{paths are quasi-geodesic, so fellow travel. Write } w = xy, \quad w' = xy', \quad \text{for } y, y' \text{ of bounded length.}
\]

\[x^{-1}gx \text{ short group element, so conjugate by short } z \text{ to } g.
\]

\[x^{-1}gx = zg^{-1} \Rightarrow g(xz) = (xz)g \Rightarrow x \text{ close to } C(g).
\]
Random Heegaard splittings.

Theorem [M]: The probability that the splitting distance of $M(w_n)$ is at most $K$ tends to zero as $n$ tends to infinity.

Requires support of $\mu$ generates a subgroup which is dense in the boundary.

Given $S$ as the boundary of a handlebody $H$, the disc set $\Delta$ is the collection of simple closed curves which bound discs in $H$. 
A Heegaard splitting has two handlebodies, with disc sets $\Delta$ and $w_n\Delta$.

Splitting distance: minimum distance between $\Delta$ and $w_n\Delta$ in $C(S)$.

[T. Kobayashi; Hempel] If the splitting distance is more than two, then $M$ is irreducible, atoroidal and not Seifert fibered.

[Perelmann] Geometrization $\Rightarrow M$ is hyperbolic.

Corollary: Probability $M(w_n)$ is hyperbolic tends to 1 as $n \to \infty$. 
[Kerckhoff] Limit set of $\triangle$ has harmonic measure zero.


Need to understand (joint) distribution of attracting and repelling endpoints.
If $g$ is pseudo-Anosov let $\lambda^+(g)$ be the attracting fixed point and let $\lambda^-(g)$ be the repelling fixed point.

Define $\lambda_n : G^\mathbb{Z}_+ \to \mathcal{F}_{\text{min}} \times \mathcal{F}_{\text{min}} \cup \emptyset$ by $\omega \mapsto (\lambda^+(w_n(\omega)), \lambda^-(w_n(\omega)))$ if $w_n(\omega)$ is pseudo-Anosov.

Claim: $\lambda_n \to \nu \times \tilde{\nu}$ as $n \to \infty$.

Reflected harmonic measure $\tilde{\nu}$ is harmonic measure determined by the random walk generated by the reflected measure $\tilde{\mu}(g) = \mu(g^{-1})$.

Halfspace: $H(1, x) = \{ y \in \hat{G} \mid \hat{d}(y, x) \leq \hat{d}(y, 1) \}$. 
If the translation length of $g$ is bigger than $K(\delta)$, then $\lambda^+(g) \in H(1, g)$, and $\lambda^-(g) \in H(1, g^{-1})$.

So $\lambda_n \sim (w_n, w_n^{-1})$. 
$H(1, w_n)$

$$P(w_{2n}(\omega) \in H(1, w_n(\omega))) \to 1 \text{ as } n \to \infty.$$  

$$P(w_{2n}^{-1}(\omega) \in H(1, w_{2n}^{-1}w_n(\omega))) \to 1 \text{ as } n \to \infty.$$  

So $(w_{2n}, w_{2n}^{-1}) \sim (w_n, w_{2n}^{-1}w_n)$.

If $w_{2n} = s_1 \ldots s_n s_{n+1} \ldots s_{2n}$, then $w_n = s_1 \ldots s_n$ and $w_{2n}^{-1}w_n = s_{2n}^{-1} \ldots s_{n+1}^{-1}$, are independent.