MATH 020 – Factoring

Fall 2009

Prime Numbers

the first eight prime numbers are: `1, 2, 3, 5, 7, 11, 13, 17`
where a prime number is a number that is only divisible by 1 and itself.
When we say "divisible" here, we mean that the result of the division is a whole number *only*.
for example, the number seven is divisible by 1 and by 7 only:

$$\frac{7}{1} = 7 \text{ and } \frac{7}{7} = 1$$

try dividing 7 by any other number and the result is NOT a whole number:

$$\frac{7}{6}, \quad \frac{7}{5}, \quad \frac{7}{4}, \quad \frac{7}{3} \quad \text{and} \quad \frac{7}{2} \text{ are not whole numbers}$$

Factors

When a number is expressed as a product of two or more numbers, those numbers are called *Factors*.
For example

$$30 = 5 \times 6$$

We say that " 5 and 6 are factors of 30".
If we further break down the factors until all the factors are prime numbers, we say that we have the **Prime Factors** of that number.

$$30 = 5 \times 6 = 5 \times 3 \times 2$$

The prime factors of 30 are 5, 3 and 2.

Greatest Common Factor or "GCF"

When we compare the factors of two numbers to see what prime factors they have in common, we say that we have found their "common prime factors", or, prime factors that they have in common. The product of these factors is called the **Greatest Common Factor, or GCF for short.**
Example:

$$30 = (5)(3)(2)$$
$$75 = (5)(5)(3)$$

The prime numbers 5 and 3 are both factors that are common to 30 and 75. The product of these factors is $3 \times 5 = 15$. Thus, the "**G**reatest **C**ommon **F**actor" of 30 and 75 is 15. To check this result, divide 15 into 30 and then into 75, and verify that in each case the result is a whole number.

$$\frac{30}{15} = 2 \quad \text{and} \quad \frac{75}{15} = 5$$

In each case, the quotients are whole numbers, which confirms that 15 is the GCF.

GCF of Variable Terms

The GCF of variable terms is determined as follows:
To find the the GCF of x^a and x^b
 if $a < b$ then the gcf is x^a
 otherwise
 if $b < a$ then the gcf is x^b
In other words, choose the lesser exponent.
Examples:

1. Find the GCF of x^3 and x^2
   the answer is: x^2
2. Find the GCF of $x^3y^5z^7$ and x^2y^6z
   the answer is: x^2y^5z
3. Find the GCF of $a^2b^6c^2$ and ab^6c^4
   the answer is: ab^6c^2

GCF of Variable Terms with Coefficients other than One
Examples:

1. Find the GCF of $12x^3$ and $18x^2$
   the answer is: $6x^2$
2. Find the GCF of $45x^3y^5z^7$ and $60x^2y^6z$
   the answer is: $15x^2y^5z$
3. Find the GCF of $9a^2b^6c^2$ and $12ab^6c^4$
   the answer is: $3ab^6c^2$

Factoring Binomials
To factor a binomial, find the GCF of the two terms, then divide the GCF into the binomial. The result is multiplied by the GCF.
Examples:

1. Factor $12x^3 + 18x^2$
   we know that the GCF is $(6x^2)$ from the above example, thus....

   $$(6x^2)\frac{12x^3 + 18x^2}{6x^2} = (6x^2)(2x + 3)$$

   To check your work, apply the distributive property by multiplying through:

   $$(6x^2)(2x + 3) = (6x^2)(2x) + (6x^2)(3) = 12x^3 + 18x^2$$

   This just brought us back to the original binomial. From this we see that *factoring* and *multiplying through (using the distributive property)* are opposite operations.

2. Factor $45x^3y^5z^7 + 60x^2y^6z$

$$(15x^2y^5z)\frac{45x^3y^5z^7 + 60x^2y^6z}{(15x^2y^5z)} = (15x^2y^5z)(3xz^6 + 4y)$$

3. Factor $9a^2b^6c^2 + 12ab^6c^4$

$$(3ab^6c^2)\frac{9a^2b^6c^2 + 12ab^6c^4}{(3ab^6c^2)} = (3ab^6c^2)(3a + 4c^2)$$

Factoring Trinomials with leading Coefficient of One

A trinomial with a leading coefficient of one looks like this: $x^2 + bx + c$

Many (but not all) trinomials in this form can be expressed as a product of two binomials. The strategy is as follows:

$x^2 + bx + c$.

Start by breaking down c into all possible pairs of factors.

example: $x^2 - 5x + 6$. c is 6 and b is -5.

What are all possible factors?

$6 = 6 \times 1$	$6 = 3 \times 2$
$6 = (-6) \times (-1)$	$6 = (-3) \times (-2)$

Then see which of these pairs of numbers adds up to b. (b is -5)

Try 6 and 1: $6 + 1 \neq -5$ no good

Try -6 and -1: $-6 + (-1) \neq -5$ no good

Try 3 and 2: $3 + 2 \neq -5$ no good

$-3 + (-2) = -5$. Yes. So we choose -3 and -2

Finally, just plug these values into the binomials: $(x - 3)(x - 2)$ is the answer. Check your results by applying FOIL to the binomial: $(x - 3)(x - 2) = x^2 - 3x - 2x + 6 = x^2 - 5x + 6$. True.

Examples:

1. Factor $x^2 + 3x + 2$
   start by factoring 2. The only pairs are $2 = (2)(1)$ and $2 = (-2)(-1)$
   Then see which of these pairs of numbers adds up to b. (b is 3)
   $-2 + (-1) \neq 3$
   $2 + 1 = 3$. So we choose 1 and 2.
   answer: $(x + 1)(x + 2)$
   to check the answer, use FOIL and see if the original trinomial is the result:
   $(x + 1)(x + 2) = x^2 + 2x + x + 2 = x^2 + 3x + 2$. True.

2. Factor $x^2 + 12x + 27$
   $27 = (9)(3)$ and $9 + 3 = 12$
   answer: $(x + 3)(x + 9)$

3. Factor $x^2 - 12x + 27$
   $27 = (-9)(-3)$ and $(-9) + (-3) = -12$
   answer: $(x-3)(x-9)$

4. Factor $x^2 + 2x - 8$
   $-8 = (4)(-2)$ and $(4) + (-2) = 2$
   answer: $(x+4)(x-2)$

5. Factor $x^2 - 4x - 12$
   $-12 = (-6)(2)$ and $(-6) + (2) = -4$
   answer: $(x+2)(x-6)$

6. Factor $x^2 - x - 1$
   cannot be expressed as a product of two binomials!

Solving Quadratic Equations
When we have a trinomial like $ax^2 + bx + c$, and we that it is equal to zero, like this $ax^2 + bx + c = 0$, it is called a **Quadratic Equation**. There will be up to two values of x that make this equation true. In order to solve for x, we need first to factor the trinomial. Let's see an example:

Solve the equation $x^2 - 8x + 15 = 0$
first, factor $x^2 - 8x + 15$
$x^2 - 8x + 15 = (x-5)(x-3)$
Therefore, $x^2 - 8x + 15 = 0$ is the same as $(x-5)(x-3) = 0$
$(x-5)(x-3) = 0$ implies that $(x-5) = 0$ or $(x-3) = 0$. So we solve both of these

$$(x-5) = 0$$
$$(x-5) + 5 = 0 + 5$$
$$x = 5$$

so x=5 is one solution. Now find the other solution:

$$(x-3) = 0$$
$$(x-3) + 3 = 0 + 3$$
$$x = 3$$

so x=3 is the other solution.
Thus, the solutions of $x^2 - 8x + 15 = 0$ is $x = 3$ and $x = 5$

The Difference of Two Squares

In a previous lesson, we found that the product of two binomials of the form $(x+a)(x-a)$ is equal to x^2-a^2, where x^2-a^2 is called the "**difference of two squares**."

$$(x+a)(x-a) = x^2 - a^2$$

Commit this equation to memory, and you can easily convert from one form to the other.
Examples:

1. Convert x^2-9 to a product of two binomials
   $x^2-9 \;=\; x^2-3^2 \;=\; (x+3)(x-3)$
   answer: $(x+3)(x-3)$

2. Convert x^2-16
   $x^2-16 \;=\; x^2-4^2 \;=\; (x+4)(x-4)$

3. Convert x^2-36
   $x^2-36 \;=\; x^2-6^2 \;=\; (x+6)(x-6)$

4. Convert $4x^2-16$
   $4x^2-16 \;=\; (2x)^2-(4)^2 \;=\; (2x+4)(2x-4)$

5. Convert $25x^2-36$
   $25x^2-36 \;=\; (5x)^2-6^2 \;=\; (5x+6)(5x-6)$