EXAMPLE 4 A bullet is fired from the ground at an angle of 60° above the horizontal. What initial speed $v_0$ must the bullet have in order to hit a point 500 ft high on a tower located 800 ft away (ignoring air resistance)?

Solution Place the gun at the origin, and let $\mathbf{r}(t)$ be the position vector of the bullet (Figure 3).

Step 1. Use Newton's Law.
Gravity exerts a downward force of magnitude $mg$, where $m$ is the mass of the bullet and $g = 32 \text{ ft/s}^2$. In vector form,

$$\mathbf{F} = (0, -gm) = m(0, -g)$$

In this case, Newton's Second Law $\mathbf{F} = m\mathbf{r}''(t)$ reduces to $\mathbf{r}''(t) = (0, -g)$. We determine $\mathbf{r}(t)$ by integrating twice:

$$\mathbf{r}'(t) = \int_0^t \mathbf{r}''(u) \, du = \int_0^t (0, -32) \, du = (0, -32t) + \mathbf{v}_0$$

$$\mathbf{r}(t) = \int_0^t \mathbf{r}'(u) \, du = \int_0^t ((0, -32u) + \mathbf{v}_0) \, du = (0, -16t^2) + t\mathbf{v}_0 + \mathbf{r}_0$$

Here, $\mathbf{r}_0$ is the initial position and $\mathbf{v}_0$ is the initial velocity. By our choice of coordinates, $\mathbf{r}_0 = 0$. The initial velocity $\mathbf{v}_0$ has unknown length $v_0$, but we know that it points in the direction of the unit vector $(\cos 60^\circ, \sin 60^\circ)$. Therefore,

$$\mathbf{v}_0 = v_0(\cos 60^\circ, \sin 60^\circ) = v_0 \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

$$\mathbf{r}(t) = (0, -16t^2) + t\mathbf{v}_0 \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

Step 2. Solve for $v_0$.

The position vector of the point on the tower is $(800, 500)$, so the bullet will hit a point on the tower 500 ft high if there exists a time $t$ such that

$$\mathbf{r}(t) = (0, -16t^2) + t\mathbf{v}_0 \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) = (800, 500)$$

Equating components, we obtain the equations

$$\frac{1}{2}tv_0 = 800, \quad -16t^2 + \frac{\sqrt{3}}{2}tv_0 = 500$$

The first equation yields $t = \frac{1.600}{v_0}$. Now substitute in the second equation and solve:

$$-16 \left( \frac{1.600}{v_0} \right)^2 + \frac{\sqrt{3}}{2} \left( \frac{1.600}{v_0} \right) v_0 = 500$$

$$\left( \frac{1.600}{v_0} \right)^2 = \frac{800\sqrt{3} - 500}{16}$$

$$v_0^2 = \frac{16(1.600)^2}{800\sqrt{3} - 500}$$

$$v_0 = \frac{6.400}{\sqrt{800\sqrt{3} - 500}} \approx 215 \text{ ft/s}$$