Calculus III (Math 233) Exam 3

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Justify answers and show all work for full credit.

NAME: ______________________________

Problem 1. Let $C$ be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(1, 3)$. Use Green’s Theorem to evaluate

$$\int_C \sqrt{1 + x^3} \, dx + 2xy \, dy$$

Problem 2. Let $F(x, y, z) = (ze^{xz}, 0, xe^{xz})$. Let $C$ be one turn of the helix,

$$C = \{ (\cos t, \sin t, t) \mid 0 \leq t \leq 2\pi \}.$$

(a) Find $f(x, y, z)$ such that $F = \nabla f$.

(b) Compute $\int_C F \cdot ds$.

Problem 3. Let $F(x, y, z) = (y^2, x, z^2)$. Let $S$ be the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, with normal oriented upward. Verify that Stokes’ Theorem is true in this case by directly evaluating both integrals.

Problem 4. Let $E$ be the solid cylinder $x^2 + y^2 \leq 1$; $0 \leq z \leq 3$. Let $F(x, y, z) = (x, y, -z)$.

(a) Directly evaluate the surface integral $\int_{\partial E} F \cdot dS$.

Note: $\partial E$ consists of the cylindrical side as well as the flat top and bottom. It may help to parametrize the side by $T(\theta, z) = (\cos \theta, \sin \theta, z)$.

(b) Verify the answer above by applying one of our theorems.

Problem 5. An open bottle $B$ lies on the $xy$-plane. Its volume is $750 \text{ ml}$. Its lip (or boundary) is the circle $\{ x^2 + (z-1)^2 = 1; y = 10 \}$. Let $F(x, y, z) = (2x + y^2, 3, x^2 + 4)$. Compute $\int_B F \cdot dS$. 
**Problem 6.** Suppose that $F$ is a vector field in $\mathbb{R}^3$ that is everywhere perpendicular to a surface $S$ with boundary $C$. Show that
\[ \iint_S (\nabla \times F) \cdot dS = 0. \]

**Problem 7. (Bonus)** If $C$ is the ellipse $x^2 + 4y^2 = 4$ oriented counterclockwise, compute (and justify)
\[ \int_C \frac{-y \, dx + (x - 1) \, dy}{(x - 1)^2 + y^2}. \]