Solutions to Sample problems for Final for Math 233

• Please attempt the problems before looking at the solutions.
• Please email me if you find any mistakes or typos.

1. (a) \[
\int_0^1 \int_0^{1-y^2} xy + 2x + 3y \, dx \, dy = 41/30
\]
(b) \[
\int_0^1 \int_0^{x^2} xe^y \, dy \, dx = (e - 2)/2
\]
(c) \[
\int_{-2}^2 \int_{x^2-3}^{5-x^2} xy \, dy \, dx = 0
\]

2. (a) \[
\int_0^{\pi/3} \int_0^{r^4} r \, dr \, d\theta = 81\pi/5
\]
(b) \[
\int_{-\pi/2}^{\pi/2} \int_0^{r^3} r \, dr \, d\theta = 4\pi
\]

3. \[
\int_0^{1/2} \int_0^{2x} e^{-x^2} \, dy \, dx = \int_0^{1/2} 2x \, e^{-x^2} \, dx = 1 - e^{-1/4}
\]

4. \[
\int_0^r \int_0^\sqrt{1+4r^2} r \, dr \, d\theta = (\pi/24)(65^{3/2} - 1)
\]

5. (a) \[
\int_0^2 \int_0^1 \int_0^1 x^2 \, dz \, dy \, dx = 64/5
\]
(b) \[
\int_0^1 \int_0^{-2-x} \int_0^{-2-y-2x} y \, dz \, dy \, dx = 1/3
\]

6. (a) \[
\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^3 \, dz \, dr \, d\theta = \pi/6
\]
(b) \[
\int_0^{\pi/2} \int_0^{2\pi} \int_0^1 \rho^8 \cos^5 \phi \sin \phi \, d\rho \, d\theta \, d\phi = \pi/27
\]

7. (a) Cylindrical. \[
\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{64-4r^2}} r \, dr \, d\theta
\]
(b) Cylindrical. \[
\int_0^{\pi/4} \int_0^1 \int_0^1 r \, dz \, dr \, d\theta
\]
(c) Spherical. \[
\int_0^{\pi/4} \int_0^{2\pi} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]
(d) Cylindrical. \[
\int_0^{2\pi} \int_0^1 \int_0^{r^2-1} r \, dz \, dr \, d\theta
\]
(e) Rectangular. \[ \int_0^2 \int_0^{2-x} \int_{4-2x-2y}^{4} dz \, dy \, dx \]

(f) Spherical. \[ \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \]

8. (a) \[ \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \pi/3 \]
(b) \[ \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\cos \phi} 2\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = 8\pi/4. \]
This is twice the volume of sphere of radius 1 centered at (0,0,1).

9. (a) Let \( x/2 = u \) and \( y/3 = v \), solve for \( u \) and \( v \) to get the transformation \( T(u,v) = (2u,3v) \). \[ \int_S (2u + 3v)6 \, du \, dv \] where \( S \) is the unit disk. Now use polar coordinates to get \[ 6 \int_0^{2\pi} \int_0^1 (2r \cos \theta + 3r \sin \theta) \, r \, dr \, d\theta = 0 \]
(b) Let \( T(u,v) = (u-v,2u-v) \). Since \( u = y-x \), we get \( 0 \leq u \leq 2 \). Since \( v = y-2x \), we get \( -1 \leq v \leq 0 \). Compute \( \frac{\partial(x,y)}{\partial(u,v)} = 1. \)
\[ \int_0^2 \int_{-1}^0 u^2 \, dv \, du = 8/3 \]