Linear Algebra (Math 338) Sample Midterm Exam 2

Date: November 22, 2005 Professor Ilya Kofman

1. Justify answers and show all work for full credit, except for Problem 1.
2. No *symbolic* calculators allowed on this exam.
3. Answer the questions in the space provided on the question sheet. If you run out of room for an answer, continue on back of the page.

NAME: ____

1. ___________

2. ___________

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Σ ___________

Problem 1. *(10 pts.) SHORT ANSWERS – NO PARTIAL CREDIT*

(a) Circle one: *Every spanning set of $\boldsymbol{R}^n$ has AT LEAST / AT MOST n vectors.*

(b) Circle one: *For $A_{m\times n}$, if rank$(A) < n$, then $Ax = 0$ has 0 / 1 / ∞-many nontrivial solutions.*

(c) *If the columns of $A_{n\times n}$ are an orthogonal set, then what are the possible values for rank(A)?*

(d) *If V has basis S, and T is obtained from S by the Gram-Schmidt process, what are the properties of T that are possibly different from S?*

(e) *Let A be a 4×3 matrix such that the sum of its columns equals 0. What is the largest possible value for the row rank of A?*

(a) AT LEAST n vectors, cannot span $\mathbf{R}^n$ if fewer vectors.

(b) Rank$(A) < n$ implies that rref(A) has mth row $= 0$, so x_m can be any real number; i.e., ∞-many nontrivial solutions.

(c) Every orthogonal set of vectors is linearly independent, so the columns of A are linearly independent. So these n linearly independent vectors are a basis for Im(A), which must be all of $\mathbf{R}^n$; i.e., rank$(A) = n$.

(d) T is an orthonormal basis. If S was already orthonormal, then $T = S$, but otherwise at least some vectors are different.

(e) Row rank = column rank, so both must be ≤ 3. But there is at least one relation among the columns, so column rank$(A) \leq 2$, so row rank$(A) \leq 2$.

Problem 2.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

Find a subset of S that is a basis for $V = span(S)$.

Let A be the 4×4 matrix with the vectors of S as its columns. This question is equivalent to: Which columns of A are a basis for $\mathrm{Im}(A)$?

$$\mathrm{rref}(A) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the first three columns of A are a basis for $\mathrm{Im}(A)$, so take the first three vectors of S.

Problem 3. *Let* $S = \{u_1, u_2, u_3\}$ *be a basis for* $\boldsymbol{R}^3$*, where*

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) *Find the coordinate vector of* $v = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$ *with respect to the basis* S*.*

(b) *If we start the Gram-Schmidt Process with* $v_1 = u_1$*, what is the second vector* v_2*?*

(a)

$$\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

To solve,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 5 \\ 1 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

(b)

$$\text{proj}_{u_2}(v_1) = \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = u_2 - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Now, normalize v_1 and v_2:

$$\tilde{v_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \tilde{v_2} = \frac{\sqrt{3}}{\sqrt{2}} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Problem 4.

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 & 6 \\ 2 & 1 & 1 & 1 & 3 \\ -1 & 4 & 10 & 1 & 12 \end{bmatrix}$$

(a) *Find the rank and nullity of A. Justify!*

(b) *Find a basis for the orthogonal complement of the null space of A.*

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 1 & 7/3 & 1/3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Rank$(A) = \#$ leading 1's $= 2$.
Nullity$(A) = n - \text{rank}(A) = 5 - 2 = 3$.

(b) Ker(A) $\perp$row(A). So basis for row space(A) = first two rows of rref(A), the rows with the leading 1's.

Problem 5. *Let P_4 be the vector space of polynomials of degree ≤ 4.*

(a) *Write down a basis for P_4.*

(b) *Is the set $\{t^4+1,\ t^3+t,\ t^2\}$ linearly independent? Justify.*

(c) *What is the orthogonal complement to P_3 in P_4?*

(a) $\{1,\ t,\ t^2,\ t^3,\ t^4\}$

(b)

$$c_1(t^4+1)+c_2(t^3+t)+c_3(t^2)=0$$
$$c_1t^4+c_2t^3+c_3t^2+c_2t+c_1=0$$

So all $c_i=0$. Yes, this set is linearly independent.

(c) As an element of P_4, every $f \in P_3$ is of the form $\begin{bmatrix} * \\ * \\ * \\ * \\ 0 \end{bmatrix}$

So $P_3^\perp = \text{span} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \text{span}(t^4) = \{ct^4 | c \in \mathbf{R}\}$

Problem 6. *Let S and T be bases for $\boldsymbol{R}^2$, where* $S = \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$.

(a) *Find the coordinate vector of* $v = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$ *with respect to S.*

(b) *Find T, given that* $P_{S \leftarrow T} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

(a)

$$\begin{bmatrix} -1 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$[v]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(b)

$$[P_{S \leftarrow T}] \begin{bmatrix} 1 \\ 0 \end{bmatrix}_T = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_S = 1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$[P_{S \leftarrow T}] \begin{bmatrix} 0 \\ 1 \end{bmatrix}_T = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}_S = -2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

Therefore, $T = \left\{ \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ -4 \end{bmatrix} \right\}$.

Problem 7. *Let* $L : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ *be defined by* $L(x, y) = (x - 2y,\ x + 2y)$.
Let $S = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ *be a basis for* $\boldsymbol{R}^2$*, and let* T *be the natural basis for* $\boldsymbol{R}^2$*.*

(a) *Find the matrix for* L *with respect to* T*.*

(b) *Find the matrix for* L *with respect to* S*.*

(c) *Find the rank and nullity of* L*.*

(a) $L(1, 0) = (1, 1)$ and $L(0, 1) = (-2, 2)$. So $A = [L]_T = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$.

(b) $L(1, -1) = (3, -1)$ and $L(0, 1) = (-2, 2)$. So $B = [L]_S = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$.

(c) Since $\text{rref}(A) = \text{rref}(B) = I_2$,
$\text{rank}(L) = \text{rank}(A) = \text{rank}(B) = 2$
$\text{nullity}(L) = \text{nullity}(A) = \text{nullity}(B) = 0$