

# Asymptotic Flatness of the Weil-Petersson Metric on Teichmüller Space

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## 1 Introduction

Let  $\Sigma$  be a smooth, closed Riemann surface of genus  $g$ , with  $n$  punctures and  $3g - 3 + n > 1$ . Let  $M_{-1}$  denote the space of metrics of constant curvature  $-1$  on  $\Sigma$ . The group,  $Diff_0$ , of diffeomorphisms homotopic to the identity acts by pull back on  $M_{-1}$ , and we can define Teichmüller space  $\mathcal{T}_{g,n}$ , to be the quotient space  $M_{-1}/Diff_0(\Sigma)$ . Note that the constant  $3g - 3 + n$  is fundamental in Teichmüller theory: it is the complex dimension of the Teichmüller space  $\mathcal{T}_{g,n}$ .

The Weil-Petersson metric on Teichmüller space has been heavily studied. It is defined by the  $L^2$  inner product on the space of holomorphic

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quadratic differentials, the cotangent space of Teichmüller space. This metric has many curious properties; for example, it is a Riemannian metric with negative sectional curvature ([29] [23]), but it is incomplete ([7] [28]). Also this metric is Kähler ([2]), and there is an upper bound  $\frac{-1}{2\pi(g-1)}$  for the holomorphic sectional curvature and Ricci curvature ([29]), a result conjectured by Royden ([19]). However, there is no known negative upper bound for the sectional curvatures.

In this paper, we investigate the asymptotic behavior of the sectional curvatures of the Weil-Petersson metric on Teichmüller space. Our method is to investigate harmonic maps from a nearly noded surface to nearby hyperbolic structures. One of the difficulties in estimating Weil-Petersson curvatures is working with the operator  $D = -2(\Delta - 2)^{-1}$ , which appears in Tromba-Wolpert's curvature formula. Our approach is to study the Hopf differentials associated to harmonic maps and the analytic formulas resulting from the harmonicity of the maps. There is a natural connection between the operator  $D$  and the local variations of the energy of a harmonic map between surfaces ([25]). From this connection, we estimate the solutions to some ordinary differential equations to derive our curvature estimates. Our estimates imply that even though the sectional curvatures are negative, they are not staying away from zero. More specifically, we prove our main theorem:

**Theorem 1.** *If the complex dimension of Teichmüller space  $\mathcal{T}$  is greater than 1, then the sectional curvature of the Weil-Petersson metric is not pinched from above by any negative constant, i.e., there is no negative upper bound for the Weil-Petersson sectional curvature.*

We remark that this asymptotic flatness results from pinching two independent closed geodesics on the surface. We will discuss more asymptotic flatness in an upcoming paper. The proof of the main theorem provides a quantitative bound for the sectional curvature, namely,

**Theorem 2.** *If the complex dimension of Teichmüller space  $\mathcal{T}$  is greater than 1, and let  $l$  be the length of shortest geodesic along a path to a boundary point in Teichmüller space, then there exists a family of tangent planes with Weil-Petersson sectional curvature of the order  $O(l)$ .*

Recently Brock and Farb ([5]) showed that the Weil-Petersson metric on  $\mathcal{T}$  is Gromov hyperbolic if and only if  $\dim_{\mathbb{C}}(\mathcal{T}) \leq 2$ . They pointed out that if the Weil-Petersson metric on  $\mathcal{T}$  had curvature pinched from above by a negative constant, the Gromov hyperbolicity in the case of the

surface being doubly-punctured torus or 5-punctured sphere would be an immediate consequence of the comparison theorems. In the end of their paper, they asked the following question: if  $\text{int}(\Sigma)$  is homeomorphic to a doubly-punctured torus or 5-punctured sphere, are the sectional curvatures of the Weil-Petersson metric bounded away from zero ([5])? Our paper gives a negative answer to this question.

Naturally associated to a harmonic map  $w : (\Sigma, \sigma|dz|^2) \rightarrow (\Sigma, \rho|dw|^2)$  is a quadratic differential  $\Phi(\sigma, \rho)dz^2$ , which is holomorphic with respect to the conformal structure of  $\sigma$ . This association of a quadratic differential to a conformal structure then defines a map  $\Phi : \mathcal{T} \rightarrow QD(\Sigma)$  from Teichmüller space  $\mathcal{T}$  to the space of holomorphic quadratic differentials  $QD(\Sigma)$ . This map is known to be a homeomorphism ([25]).

As an important computational tool in geometry of Teichmüller theory, the method of harmonic maps has been studied by many people. In particular, the second variation of the energy of the harmonic map  $w = w(\sigma, \rho)$  with respect to the domain structure  $\sigma$  (or image structure  $\rho$ ) at  $\sigma = \rho$  yields the Weil-Petersson metric on  $\mathcal{T}$  ([23], [25]), and one can also establish Tromba-Wolpert's curvature formula of the Weil-Petersson metric from this method ([13], see also [25]).

The moduli space of Riemann surfaces admits a compactification, known as the Deligne-Mumford compactification ([16]), and any element of the compactification divisor can be thought of as a Riemann surface with nodes, a connected complex space where points have neighborhoods complex isomorphic to either  $\{|z| < \varepsilon\}$  (regular points) or  $\{zw = 0; |z|, |w| < \varepsilon\}$  (nodes). We can think of noded surfaces arising as elements of the compactification divisor through a pinching process: fix a family of simple closed curves on  $\Sigma$  such that each component of the complement of the curves has negative Euler characteristic. The noded surface is topologically the result of identifying each curve to a point, the node ([4]).

While writing this paper, the author learned that Scott Wolpert ([30]) has obtained results related to ours.

The organization of this paper is as follows. In section 2 we give the necessary background, define our terms and introduce the notations. Section 3 is devoted to proving our main theorem. The discussion of this purpose is broken into subsections: in §3.1, we collect the local variational formulas associated with the harmonic maps, hence connect the local variations of the energy of the harmonic map to the operator  $D = -2(\Delta - 2)^{-1}$ ; in §3.2 we describe the so called "model case" of the problem; namely, we pinch two core geodesics of two cylinders into two points, and study the asymptotic behavior of the harmonic maps between cylinders. In the model

case of this problem, the surface is a pair of cylinders; in §3.3, we describe a family of asymptotically flat 2-dimensional subspaces. We also establish two ordinary differential equations, and relate the solutions of the equations to the operator  $D = -2(\Delta - 2)^{-1}$ , which leads us to estimate the curvature. We will establish the estimates of terms in the curvature formula when the surface is a pair of cylinders; in §3.4 we construct families of maps which have small tension, and are close to the harmonic maps resulting from pinching process in §3.3; finally in §3.5 we prove our main theorem using the estimates in §3.3 and the construction in §3.4. In section 4, we comment on the case when the surface has finitely many punctures.

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## 2 Notations and Background

Recall that  $\Sigma$  is a fixed, oriented, smooth surface of genus  $g \geq 1$ , and  $n \geq 0$  punctures where  $3g - 3 + n > 1$ . We denote hyperbolic metrics on  $\Sigma$  by  $\sigma|dz|^2$  and  $\rho|dw|^2$ , where  $z$  and  $w$  are conformal coordinates on  $\Sigma$ . By the uniformization theorem, the set of all similarly oriented hyperbolic structures  $M_{-1}$  can be identified with the set of all conformal structures on  $\Sigma$  with the given orientation. Equivalently, this is the same as the set of all complex structures on  $\Sigma$  with the given orientation. And Teichmüller space  $\mathcal{T}$  is defined to be

$$\mathcal{T} = M_{-1}/Diff_0(\Sigma)$$

It is well-known that Teichmüller space  $\mathcal{T}$  has a complex structure ([2]) and the cotangent space at a point  $\Sigma \in \mathcal{T}$  is the space of holomorphic quadratic differentials  $\Phi dz^2$  on  $\Sigma$ . On  $\Sigma$ , there is a natural pairing of quadratic differentials and Beltrami differentials  $\mu(z)\frac{d\bar{z}}{dz}$  given by

$$\langle \mu, \Phi \rangle = \operatorname{Re} \int_{\Sigma} \mu(z)\Phi(z)dzd\bar{z}$$

The tangent space at  $\Sigma$  is the space of Beltrami differentials modulo the trivial ones which are ones such that  $\langle \mu, \Phi \rangle = 0$ . The Weil-Petersson metric on  $\mathcal{T}$  is obtained by duality from the  $L^2$ -inner product on  $QD(\Sigma)$

$$\langle \phi, \psi \rangle = \int_{\Sigma} \frac{\phi\bar{\psi}}{\sigma} dzd\bar{z}$$

where  $\sigma dzd\bar{z}$  is the hyperbolic metric on  $\Sigma$ .

For a Lipschitz map  $w : (\Sigma, \sigma|dz|^2) \rightarrow (\Sigma, \rho|dw|^2)$ , we define the energy density of  $w$  at a point to be

$$e(w; \sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2 + \frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2$$

and the total energy  $E(w; \sigma, \rho) = \int_{\Sigma} e(w; \sigma, \rho) \sigma dzd\bar{z}$ .

A critical point of  $E(w; \sigma, \rho)$  is called a harmonic map; it satisfies the Euler-Lagrange equation, namely,

$$w_{z\bar{z}} + \frac{\rho_w}{\rho} w_z w_{\bar{z}} = 0.$$

The Euler-Lagrange equation for the energy is the condition for the vanishing of the *tension*, which is, in local coordinates,

$$\tau(w) = \Delta w^\gamma + \Gamma_{\alpha\beta}^\gamma w_i^\alpha w_j^\beta = 0$$

It is well-known ([3] [9] [11] [22] [20]) that given  $\sigma, \rho$ , there exists a unique harmonic map  $w : (\Sigma, \sigma) \rightarrow (\Sigma, \rho)$  homotopic to the identity of  $\Sigma$ , and this map is in fact a diffeomorphism.

Thus we have a holomorphic quadratic differential (also called a Hopf differential)  $\Phi dz^2 = \rho w_z \bar{w}_z dz^2$ , and evidently

$$\Phi = 0 \Leftrightarrow w \text{ is conformal} \Leftrightarrow \sigma = \rho.$$

where  $\sigma = \rho$  means that  $(\Sigma, \sigma)$  and  $(\Sigma, \rho)$  are the same point in Teichmüller space  $\mathcal{T}$ . This describes  $\Phi$  as a well-defined map  $\Phi : \mathcal{T} \rightarrow QD(\Sigma)$  from Teichmüller space  $\mathcal{T}$  to the space of Hopf differentials  $QD(\Sigma)$ . This map is a homeomorphism ([25]). Also note that the map  $w$  can be extended to surfaces with finitely many punctures.

We define two auxiliary functions as following:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}(\sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2 \\ \mathcal{L} &= \mathcal{L}(\sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2 \end{aligned}$$

Hence the energy density is  $e = \mathcal{H} + \mathcal{L}$ . Many aspects connected with a harmonic map between Riemann surfaces can be written in terms of  $\mathcal{H}$  and  $\mathcal{L}$  (and  $\Phi$ ). We denote on  $(\Sigma, \sigma|dz|^2)$

$$\Delta = \frac{4}{\sigma} \frac{\partial^2}{\partial z \partial \bar{z}}, K(\rho) = -\frac{2}{\rho} \frac{\partial^2}{\partial w \partial \bar{w}} \log \rho, K(\sigma) = -\frac{2}{\sigma} \frac{\partial^2}{\partial z \partial \bar{z}} \log \sigma,$$

where  $\Delta$  is the Laplacian, and  $K(\rho)$ ,  $K(\sigma)$  are curvatures of the metrics  $\rho$  and  $\sigma$ , respectively.

The Euler-Lagrange equation gives

$$\Delta \log \mathcal{H} = -2K(\rho)\mathcal{H} + 2K(\rho)\mathcal{L} + 2K(\sigma).$$

When we restrict ourselves to the situation  $K(\sigma) = K(\rho) = -1$ , we will have the following facts ([25]):

- $\mathcal{H} > 0$ ;
- The Beltrami differential  $\mu = \frac{w_{\bar{z}}}{w_z} = \frac{\bar{\Phi}}{\sigma\mathcal{H}}$ ;
- $\Delta \log \mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2$ .

### 3 Weil-Petersson Sectional Curvature

In this section, we will prove the main theorem of this paper, namely,

**Theorem 1.** *If the complex dimension of Teichmüller space  $\mathcal{T}$  is greater than 1, then the sectional curvature of the Weil-Petersson metric is not pinched from above by any negative constant, i.e. there is no negative upper bound for the Weil-Petersson sectional curvature.*

Combined with the fact that there is no lower bound for the sectional curvature ([21]), We have

**Corollary 3.** *There are no negative bounds for the sectional curvature of the Weil-Petersson metric when  $\dim_{\mathbb{C}}(\mathcal{T}) > 1$ .*

The rest of this paper will be devoted to proving our main theorem. The discussion will be broken into 5 subsections. In §3.1, we collect some variational formulas we will need in the rest of discussion; in §3.2, we mostly describe the “model case”, i.e. we consider a family of perturbations of the identity map between a pair of cylinders; in §3.3, we describe a family of subplanes of the tangent space of Teichmüller space  $\mathcal{T}_{g,2}$ , and estimate terms in the curvature formula when the surface is a pair of cylinders; in §3.4, we construct  $C^{2,\alpha}$  maps between surfaces and show that the constructed maps have small tension and are close to the harmonic maps we get from the pinching process; in §3.5, we prove the main theorem.

#### 3.1 Local Variation

We consider a family of harmonic maps  $w(t)$  for  $t$  small, where  $w(0) = \text{id}$ , the identity map. Denote by  $\Phi(t)$  the family of Hopf differentials determined by  $w(t)$ . We can rewrite the equation  $\Delta \log \mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2$  as

$$\Delta \log \mathcal{H}(t) = 2\mathcal{H}(t) - \frac{2|\Phi(t)|^2}{\sigma^2 \mathcal{H}(t)} - 2$$

For this equation, we see that the maximum principle will force all the odd order  $t$ -derivatives of  $\mathcal{H}(t)$  to vanish, since the above equation only depends on the modulus of  $\Phi(t)$  and not on its argument ([25]), and  $\mathcal{H}(t)$  is real-analytic in  $t$  ([26]).

Wolf computed the  $t$ -derivative of various geometric quantities associated with the harmonic maps, and we collect these local variational formulas into our next lemma:

**Lemma 4.** ([25]) *For the above notations, we have*

- $\mathcal{H}(t) \geq 1$ , and  $\mathcal{H}(t) \equiv 1 \Leftrightarrow t = 0$ ;
- $\dot{\mathcal{H}}(t) = \partial/\partial t^\alpha|_0 \mathcal{H}(t) = 0$ ;
- $\dot{\mu} = \partial/\partial t^\alpha|_0 \mu(t) = \bar{\Phi}_\alpha/\sigma$ ;
- $\ddot{\mathcal{H}}(t) = \frac{\partial^2}{\partial t^\alpha \partial t^\beta}|_0 \mathcal{H}(t) = D \frac{\Phi_\alpha \Phi_\beta}{\sigma^2}$ , where  $D = -2(\Delta - 2)^{-1}$ . Evidently  $D$  is a self-adjoint compact integral operator with a positive kernel, and it is the identity on constant functions.

With this lemma, we obtain a partial differential equation about  $\ddot{\mathcal{H}}(t)$ :

$$(\Delta - 2)(\ddot{\mathcal{H}}(t)) = -2 \frac{\Phi_\alpha \Phi_\beta}{\sigma^2}$$

### 3.2 Harmonic Maps Between Cylinders

For the sake of simplicity of exposition, we assume that our surface has no punctures. We will comment on punctured case in section 4.

Before we jump into the proof of the main theorem, in this subsection, we consider the asymptotics of harmonic maps when our surface is a pair of cylinders. We consider the asymptotics of a family of perturbations of the identity map between two cylinders, and we call this is the “model case”. In the model case, the surface is a pair of cylinders  $M_0$  and  $M_1$ . In particular, consider the boundary value problem of harmonically mapping the cylinder

$$M = [l^{-1} \sin^{-1}(l), \pi l^{-1} - l^{-1} \sin^{-1}(l)] \times [0, 1]$$

with boundary identification

$$[\frac{\sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{\sin^{-1}(l)}{l}] \times \{0\} = [\frac{\sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{\sin^{-1}(l)}{l}] \times \{1\}$$

where the hyperbolic length element on  $M$  is  $l \csc(lx) |dz|$ , to the cylinder

$$N = [L^{-1} \sin^{-1}(L), \pi L^{-1} - L^{-1} \sin^{-1}(L)] \times [0, 1]$$

with boundary identification

$$[\frac{\sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{\sin^{-1}(L)}{L}] \times \{0\} = [\frac{\sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{\sin^{-1}(L)}{L}] \times \{1\}$$

where the hyperbolic length element on  $N$  is  $Lcsc(Lu)|dw|$ . Here  $l$  and  $L$  are the lengths of the simple closed core geodesics in the corresponding cylinders.

Say  $w = u + iv$  is this harmonic map between cylinders  $M$  and  $N$ , where

$$u(l, L; x, y) = u(l, L; x), v(x, y) = y.$$

The Euler-Lagrange equation becomes

$$u'' = Lcsc(Lu)(u'^2 - 1).$$

with boundary conditions  $u(\frac{\sin^{-1}(l)}{l}) = \frac{\sin^{-1}(L)}{L}$  and  $u(\frac{\pi}{2l}) = \frac{\pi}{2L}$ . Note that both  $M$  and  $N$  admit an anti-isometric reflection about the curves  $\{\frac{\pi}{2l}\} \times [0, 1]$  and  $\{\frac{\pi}{2L}\} \times [0, 1]$ .

Since the quadratic differential  $\Phi = \rho w_z \bar{w}_z = \frac{1}{4} L^2 csc^2(Lu)(u'^2 - 1)$  is holomorphic in  $M_0$ , we have

$$0 = \frac{\partial}{\partial \bar{z}}(\rho^2 w_z \bar{w}_z) = \frac{\partial}{\partial x}(\frac{1}{8} L^2 csc^2(Lu)(u'^2 - 1))$$

Therefore  $L^2 csc^2(Lu)(u'^2 - 1) = c_0(l, L)$ , where  $c_0(l, L)$  is independent of  $x$ , and  $c_0(l, l) = 0$  since  $u(x)$  is the identity map when  $L = l$ .

So we have

$$u' = \sqrt{1 + c_0(l, L)L^{-2} \sin^2(Lu)}$$

with boundary conditions  $u(\frac{\sin^{-1}(l)}{l}) = \frac{\sin^{-1}(L)}{L}$  and  $u(\frac{\pi}{2l}) = \frac{\pi}{2L}$ . So the solution to the Euler-Lagrange equation can be derived from the equation

$$\int_{L^{-1} \sin^{-1} L}^u \frac{dv}{\sqrt{1 + c_0(l, L)L^2 \sin^2(Lv)}} = x - l^{-1} \sin^{-1}(l)$$

with  $c_0(l, L)$  chosen such that  $\int_{L^{-1} \sin^{-1} L}^{\frac{\pi}{2L}} \frac{dv}{\sqrt{1 + c_0(l, L)L^2 \sin^2(Lv)}} = \frac{\pi}{2l} - l^{-1} \sin^{-1} l$ .

It is not hard to show that when  $l \rightarrow 0$ , the solution  $u(l, L; x)$  converges to a solution  $u(L; x)$  to the “noded” problem, i.e.,  $M = [1, +\infty)$ , where we require  $u(L; 1) = L^{-1} \sin^{-1}(L)$  and  $\lim_{x \rightarrow +\infty} u(L; x) = \frac{\pi}{2L}$ .

This “noded” problem has the explicit solution as following ([26])

$$u(L; x) = L^{-1} \sin^{-1} \left\{ \frac{1 - \frac{(1-L)}{(1+L)} e^{2L(1-x)}}{1 + \frac{(1-L)}{(1+L)} e^{2L(1-x)}} \right\}$$

with the holomorphic energy

$$H_0(L; x) = \frac{L^2 x^2}{4} \left[ \frac{1 + \sqrt{\frac{(1-L)}{(1+L)}} e^{L(1-x)}}{1 - \sqrt{\frac{(1-L)}{(1+L)}} e^{L(1-x)}} \right]^2$$

### 3.3 Asymptotically Flat Subplanes

Consider the surface  $\Sigma$  which is developing two nodes, i.e., we are pinching two nonhomotopic closed geodesics  $\gamma_0$  and  $\gamma_1$  on  $\Sigma$  to two points, say  $p_0$  and  $p_1$ . We denote  $M_0$  and  $M_1$  their pinching neighborhoods, i.e., two cylinders described as  $M$  above centered at  $\gamma_0$  and  $\gamma_1$ , respectively.

We define  $M(l_0, l_1)$  be the surface with two of the Fenchel-Nielsen coordinates, namely, the hyperbolic lengths of  $\gamma_0$  and  $\gamma_1$ , are  $l_0$  and  $l_1$ , respectively. When we set the length of these two geodesics equal to  $l$  simultaneously, we will have a point  $M(l) = M(l, l)$  in Teichmüller space  $\mathcal{T}_g$ . Note that as  $l$  tends to zero, the surface is developing two nodes. At this point  $M(l)$ , we look at two directions. First, we fix  $\gamma_1$  in  $M_1$  having length  $l$ , and pinch  $\gamma_0$  in  $M_0$  into length  $L = L(t)$ , where  $L(0) = l$ . So the  $t$ -derivative of  $\mu_0(t)$ , the Beltrami differential of the resulting harmonic map, at  $t = 0$  represents a tangent vector,  $\dot{\mu}_0$ , of Teichmüller space  $\mathcal{T}_g$  at  $M(l)$ ; we denote the resulting harmonic map by  $W_0(t) : M(l, l) \rightarrow M(L(t), l)$ . Then we fix  $\gamma_0$  in  $M_0$  having length  $l$ , and pinch  $\gamma_1$  in  $M_1$  into length  $L = L(t)$ , so the  $t$ -derivative of  $\mu_1(t)$  at  $t = 0$  represents another tangent vector,  $\dot{\mu}_1$ , at  $M(l)$ ; we denote the resulting harmonic map by  $W_1(t) : M(l, l) \rightarrow M(l, L(t))$ . These two tangent vectors  $\dot{\mu}_0$  and  $\dot{\mu}_1$  will span a 2-dimensional subspace of the tangent space  $T_{M(l)}\mathcal{T}_g$  to  $\mathcal{T}_g$ , hence we obtain a family,  $\Omega_l$ , of 2-dimensional subspaces of the tangent space of  $\mathcal{T}_g$ .

**Proposition 5.** *The Weil-Petersson sectional curvatures of  $\Omega_l$  tend to zero as  $l$  tends to zero.*

It is immediate that this proposition implies our main theorem.

To begin the proof of the proposition, we look at the curvature tensor of the Weil-Petersson metric, which is given by ([29])

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \left( \int_{\Sigma} D(\dot{\mu}_{\alpha}\dot{\mu}_{\bar{\beta}})\dot{\mu}_{\gamma}\dot{\mu}_{\bar{\delta}}dA \right) + \left( \int_{\Sigma} D(\dot{\mu}_{\alpha}\dot{\mu}_{\bar{\delta}})\dot{\mu}_{\gamma}\dot{\mu}_{\bar{\beta}}dA \right)$$

where  $dA$  is the area element. Then the curvature of  $\Omega$  is  $R/\Pi$ , where ([29])

$$R = R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}}$$

and

$$\begin{aligned} \Pi &= 4 \langle \dot{\mu}_0, \dot{\mu}_0 \rangle \langle \dot{\mu}_1, \dot{\mu}_1 \rangle - 2 | \langle \dot{\mu}_0, \dot{\mu}_1 \rangle |^2 - 2 \operatorname{Re}(\langle \dot{\mu}_0, \dot{\mu}_1 \rangle)^2 \\ &= 4 \langle \dot{\mu}_0, \dot{\mu}_0 \rangle \langle \dot{\mu}_1, \dot{\mu}_1 \rangle - 4 | \langle \dot{\mu}_0, \dot{\mu}_1 \rangle |^2 \end{aligned}$$

In the rest of this subsection, we will estimate terms in the above curvature formula in the model case, i.e., the surface is a pair of cylinders  $M_0$  and  $M_1$ .

We denote  $\phi_0(t)$  as the Hopf differential corresponding to the cylinder map  $w_0(t)$  in  $M_0$ , and  $\phi_1(t)$  as the Hopf differential corresponding to the cylinder map  $w_1(t)$  in  $M_1$ . Here  $w_0(t) : M_0(l) \rightarrow M_0(L(t))$  and  $w_1(t) : M_1(l) \rightarrow M_1(L(t))$  are harmonic maps described as  $w = u(x) + iy$  in the last subsection. In  $M_1$ , we still use  $\phi_0$  to denote the Hopf differential corresponding to harmonic map  $W_0(t)$ , while in  $M_0$ , we still use  $\phi_1$  as the Hopf differential corresponding to harmonic map  $W_1(t)$ . We also denote  $\mu_0$  and  $\mu_1$  as the corresponding Beltrami differentials to  $\phi_0$  and  $\phi_1$ .

We denote  $a = a(l) = l^{-1} \sin^{-1}(l)$ , and  $b = b(l) = \pi l^{-1} - l^{-1} \sin^{-1}(l)$ . And in this paper,  $A \sim B$  means  $A/C < B < CA$  for some constant  $C > 0$ .

So in  $M_0$ , as in §3.2, we choose  $L(t)$  so that  $\frac{d}{dt}|_{t=0} c_0(t) = \frac{d}{dL}|_{L=l} c_0(l, L) = 4$ . Here we recall that  $c_0(l, L(t)) = L^2 c s c^2(Lu)(u^2 - 1)$ . We notice that  $\dot{c}_0$  is never zero for all  $l > 0$ , otherwise we would have  $\dot{w}_z = 0$  as  $\dot{\phi} = \sigma \dot{w}_z$ . Hence  $w$  is a constant map by rotational invariance of the map. Thus we have  $\dot{\phi}_0 = \frac{d}{dt}|_{t=0} (\frac{1}{4} c_0(t)) = 1$  in  $M_0$ .

Also, we see that  $|\dot{\phi}_1|_{M_0} = \zeta(x, l)$  for  $x \in [a, b]$ , where  $\zeta(x, l)$  satisfies that  $\zeta(x, l) \leq C_1 x^{-4}$  for  $x \in [a, \pi/2l]$ , and  $\zeta(x, l) \leq C_1(\pi/l - x)^{-4}$  for  $x \in [\pi/2l, b]$ , and  $\zeta(x, 0)$  decays exponentially in  $[1, +\infty]$ . Here  $C_1 = C_1(l)$  is positive and bounded as  $l \rightarrow 0$ . To see this, notice that  $\dot{\phi}_1$  is holomorphic and  $|\dot{\phi}_1|$  is positive, so  $\log|\dot{\phi}_1|$  is harmonic in the cylinder  $M_0$ . Hence we can express  $\log|\dot{\phi}_1|$  in a Fourier series  $\sum a_n(x) \exp(-iny)$ , and we compute  $0 = \Delta \log|\dot{\phi}_1| = \sum (a_n'' - n^2 a_n) \exp(-iny)$ . We will see, in §3.4, that  $\dot{\phi}_1$  is close to 0 in  $M_0$ . Hence we conclude the properties  $\zeta$  has. Similarly, we have  $\dot{\phi}_1|_{M_1} = 1$  and  $|\dot{\phi}_0|_{M_1} = \zeta(x, l)$  for  $x \in [a, b]$ .

Note that these estimates imply, informally, that most of the mass of  $|\phi_0|$  resides in the thin part associated to  $\gamma_0$ , and most of the mass of  $|\phi_1|$  resides in the thin part associated to  $\gamma_1$ .

Now in  $M_0$ , the corresponding Beltrami differential is

$$\dot{\mu}_0 = \frac{d}{dt}|_{t=0} \left( \frac{w_{\bar{z}}}{w_z} \right) = \dot{\phi}_0 / \sigma$$

and  $|\dot{\mu}_0|^2|_{M_0} = |\dot{\phi}_0 / \sigma|^2|_{M_0} = l^{-4} \sin^4(lx)$ .

Also, in that same neighborhood  $M_0$ ,

$$\dot{\mu}_1 = \dot{\phi}_1 / \sigma$$

Hence

$$|\dot{\mu}_1|^2|_{M_0} = |\dot{\phi}_1 / \sigma|^2|_{M_0} = l^{-4} \sin^4(lx) \zeta^2(x, l).$$

Similarly, in  $M_1$ , we have

$$\begin{aligned} |\dot{\mu}_0|^2|_{M_1} &= l^{-4} \sin^4(lx) \zeta^2(x, l) \\ |\dot{\mu}_1|^2|_{M_1} &= l^{-4} \sin^4(lx) \end{aligned}$$

**Lemma 6.**  $1/\Pi = O(l^3)$ .

We recall that  $\Pi = 4 \langle \dot{\mu}_0, \dot{\mu}_0 \rangle \langle \dot{\mu}_1, \dot{\mu}_1 \rangle - 4 | \langle \dot{\mu}_0, \dot{\mu}_1 \rangle |^2$ , and compute the asymptotics in  $l$  of each term. Using  $|\dot{\mu}_0|^2|_{M_0} = l^{-4} \sin^4(lx)$ , and noting that  $a = a(l) = l^{-1} \sin^{-1}(l)$ , and  $b = b(l) = \pi l^{-1} - l^{-1} \sin^{-1}(l)$ , we have that

$$\begin{aligned} \langle \dot{\mu}_0, \dot{\mu}_0 \rangle|_{M_0} &= \int_{M_0} |\dot{\mu}_0|^2 \sigma dx dy \\ &= \int_0^1 \int_a^b |\dot{\mu}_0|^2 \sigma dx dy \\ &= \int_0^1 \int_a^b l^{-2} \sin^2 lx dx dy \\ &= \frac{\pi}{2} l^{-3} - l^{-3} \sin^{-1}(l) \\ &\sim l^{-3} \end{aligned}$$

And using  $|\dot{\mu}_1|^2|_{M_0} = l^{-4} \sin^4(lx) \zeta^2(x, l)$ , and  $\zeta(x, l) \leq C_1 x^{-4}$  for  $x \in [a, \pi/2l]$ , we have

$$\begin{aligned} \langle \dot{\mu}_1, \dot{\mu}_1 \rangle|_{M_0} &= \int_{M_0} |\dot{\mu}_1|^2 \sigma dx dy \\ &= 2 \int_0^1 \int_a^{\pi/2l} l^{-4} \sin^4(lx) \zeta^2(x, l) \sigma dx dy \\ &\leq 2C_1^2 \int_0^1 \int_a^{\pi/2l} l^{-2} \sin^2(lx) x^{-8} dx dy \\ &= O(1) \end{aligned}$$

Also  $\langle \dot{\mu}_0, \dot{\mu}_1 \rangle = \int_{\Sigma} \dot{\mu}_0 \dot{\mu}_1 dA$ , hence,

$$\begin{aligned} \langle \dot{\mu}_0, \dot{\mu}_1 \rangle|_{M_0} &= \int_{M_0} \dot{\mu}_0 \dot{\mu}_1 \sigma dx dy \\ &\leq C_1 \int_{M_0} l^{-2} \sin^2(lx) x^{-4} dx dy \\ &= O(1) \end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \dot{\mu}_0, \dot{\mu}_0 \rangle |_{M_1} &= O(1) \\
\langle \dot{\mu}_1, \dot{\mu}_1 \rangle |_{M_1} &\sim l^{-3} \\
\langle \dot{\mu}_0, \dot{\mu}_1 \rangle |_{M_1} &= O(1)
\end{aligned}$$

Note that  $\Pi \geq (4 \langle \dot{\mu}_0, \dot{\mu}_0 \rangle \langle \dot{\mu}_1, \dot{\mu}_1 \rangle - 4(\langle \dot{\mu}_0, \dot{\mu}_1 \rangle)^2) |_{M_0} \sim l^{-3}$ , which completes the proof of Lemma 6.

From Lemma 6, we have

$$|R|/\Pi = O(|R|/(l^{-3})) = O(|R|l^3) \quad (1)$$

Now we are left to estimate  $|R|$ . We have

**Lemma 7.**  $|R| \leq 4 \int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 \sigma dx dy$

Note that  $D = -2(\Delta - 2)^{-1}$  is self-adjoint, hence we have

$$\int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 \sigma dx dy = \int_{\Sigma} D(|\dot{\mu}_1|^2) |\dot{\mu}_0|^2 \sigma dx dy$$

Therefore,

$$\begin{aligned}
R &= R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}} \\
&= 2 \int_{\Sigma} D(\dot{\mu}_0 \dot{\mu}_1) \dot{\mu}_0 \dot{\mu}_1 \sigma dx dy + 2 \int_{\Sigma} D(\dot{\mu}_1 \dot{\mu}_0) \dot{\mu}_1 \dot{\mu}_0 \sigma dx dy \\
&\quad - \int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 \sigma dx dy - \int_{\Sigma} D(|\dot{\mu}_1|^2) |\dot{\mu}_0|^2 \sigma dx dy \\
&\quad - \int_{\Sigma} D(\dot{\mu}_0 \dot{\mu}_1) \dot{\mu}_1 \dot{\mu}_0 \sigma dx dy - \int_{\Sigma} D(\dot{\mu}_1 \dot{\mu}_0) \dot{\mu}_0 \dot{\mu}_1 \sigma dx dy \\
&= 2 \int_{\Sigma} D(\dot{\mu}_0 \dot{\mu}_1) \dot{\mu}_1 \dot{\mu}_0 \sigma dx dy - 2 \int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 \sigma dx dy
\end{aligned}$$

The last equality follows from that here  $\dot{\mu}_0$  and  $\dot{\mu}_1$  are real functions. Now from lemma 4.3 of [29], we have  $|D(\dot{\mu}_0 \dot{\mu}_1)| \leq |D(|\dot{\mu}_0|^2)|^{1/2} |D(|\dot{\mu}_1|^2)|^{1/2}$ . Then an application of Hölder inequality shows that

$$\begin{aligned}
\left| \int_{\Sigma} D(\dot{\mu}_0 \dot{\mu}_1) \dot{\mu}_1 \dot{\mu}_0 dA \right| &\leq \int_{\Sigma} |D(|\dot{\mu}_0|^2)|^{1/2} |D(|\dot{\mu}_1|^2)|^{1/2} \dot{\mu}_1 \dot{\mu}_0 dA \\
&\leq \left( \int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 dA \right)^{\frac{1}{2}} \left( \int_{\Sigma} D(|\dot{\mu}_1|^2) |\dot{\mu}_0|^2 dA \right)^{\frac{1}{2}} \\
&= \int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 \sigma dx dy
\end{aligned}$$

So we will have

$$|R| \leq 4 \int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 \sigma dx dy \quad (2)$$

which completes the proof of Lemma 7.

Therefore together with (1) and (2), to show the proposition, it is enough to show that  $\int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dx dy = o(l^{-3})$ . In fact,

**Lemma 8.**  $\int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dx dy = O(l^{-2})$

Firstly, from Lemma 4, we have

$$D(|\mu_0|^2) = -2(\Delta - 2)^{-1} \frac{|\dot{\phi}_0|^2}{\sigma^2}$$

We recall in  $M_0$ , the Hopf differential  $\phi_0$  is corresponding to the cylinder map  $w_0(t) : (M_0, \sigma) \rightarrow (M_0, \rho(t))$ , and  $|\dot{\phi}_0| = 1$ . As in section 2 and §3.1, we write  $\mathcal{H} = \frac{\rho(w(z))}{\sigma(z)} |w_z|^2$ , therefore we can write  $D(|\mu_0|^2) = \ddot{\mathcal{H}}$ . Then in the cylinder  $M_0$ , we have

$$(\Delta - 2)\ddot{\mathcal{H}} = -2 \frac{|\dot{\phi}_0|^2}{\sigma^2} = -2l^{-4} \sin^4(lx) \quad (3)$$

A maximum principle argument implies that  $\ddot{\mathcal{H}}$  is positive. This  $\ddot{\mathcal{H}}$  converges to the holomorphic energy  $\ddot{H}_0$  in the “noded” problem (of §3.2) when  $x$  is fixed but sufficiently large in  $[a, \pi/2l]$ . This convergence guarantees that  $\ddot{\mathcal{H}}$  is bounded on the compacta in  $[a, b]$ . Thus we can assume that  $A_1(l) = \ddot{\mathcal{H}}(a) = \ddot{\mathcal{H}}(l^{-1} \sin^{-1} l) = O(1) > 0$ . Then  $\ddot{\mathcal{H}}(x)$  solves the following differential equation

$$(l^{-2} \sin^2(lx))\ddot{\mathcal{H}}'' - 2\ddot{\mathcal{H}} = -2l^{-4} \sin^4(lx) \quad (4)$$

with the conditions

$$\ddot{\mathcal{H}}(l^{-1} \sin^{-1} l) = A_1(l), \ddot{\mathcal{H}}'(\pi/2l) = 0$$

Recall from §3.1 that all the odd order  $t$ -derivatives of  $\mathcal{H}(t)$  vanish. Also notice that

$$J(x) = \frac{\sin^2(lx)}{2l^4}$$

is a particular solution to equation (4). Hence we can check, by the method of reduction of the solutions, the general solution to equation (4) with the assigned conditions has the form

$$\ddot{\mathcal{H}}(l; x) = J(x) + A_2 \cot(lx) + A_3(1 - lx \cot(lx))$$

where coefficients  $A_2 = A_2(l)$  and  $A_3 = A_3(l)$  are constants independent of  $x$  and we can check, by substituting the solution into the assigned conditions, that they satisfy that

$$A_2 = \frac{\pi}{2}A_3 = O(l^{-1})$$

Noticing that  $\zeta(x, l) \leq C_1 x^{-4}$  in  $[a, \pi/2l]$ , we compute the following

$$\begin{aligned} \int_{M_0} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 \sigma dx dy &= \int_0^1 \int_a^b \ddot{\mathcal{H}}(x)(l^{-2} \sin^2(lx)) \zeta^2(x, l) dx dy \\ &\leq 2C_1^2 \left( \int_0^1 \int_a^{\pi/2l} J(x) l^{-2} \sin^2(lx) x^{-8} dx dy \right. \\ &\quad + \int_0^1 \int_a^{\pi/2l} A_2 \cot(lx) l^{-2} \sin^2(lx) x^{-8} dx dy \\ &\quad \left. + \int_0^1 \int_a^{\pi/2l} A_3 (1 - lx \cot(lx)) l^{-2} \sin^2(lx) x^{-8} dx dy \right) \\ &= O(l^{-2}) + O(l^{-2}) + O(l^{-2}) \\ &= O(l^{-2}) \end{aligned}$$

Therefore

$$\begin{aligned} \int_{M_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy &= \int_a^b \ddot{\mathcal{H}}|\dot{\mu}_1|^2 \sigma dx \\ &= O(l^{-2}) \end{aligned} \quad (5)$$

Now let us look at the term  $\int_{M_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy$ . Note that in  $M_1$ , which we identify with  $[a, b] \times [0, 1]$ , we have  $D(|\dot{\mu}_0|^2) = \ddot{\mathcal{H}}$ , here  $\mu_0(t)$  and  $\mathcal{H}$  come from the harmonic map  $W_0(t) : M(l, l) \rightarrow M(L(t), l)$ , where  $|\dot{\mu}_0|_{M_1} = l^{-2} \sin^2(lx) \zeta(x, l)$ , and  $\mathcal{H}(t)$  solves

$$(l^{-2} \sin^2(lx)) \ddot{\mathcal{H}}'' - 2\ddot{\mathcal{H}} = -2l^{-4} \sin^4(lx) \zeta^2(x, l) \quad (6)$$

with the conditions

$$\ddot{\mathcal{H}}(l^{-1} \sin^{-1} l) = B_1(l), \quad \ddot{\mathcal{H}}(\pi/l - l^{-1} \sin^{-1} l) = B_2(l).$$

Here  $B_1(l)$  and  $B_2(l)$  are positive and bounded as  $l \rightarrow 0$ , since  $\ddot{\mathcal{H}}$  converges to the holomorphic energy in the ‘‘noded’’ problem (of §3.2, when  $M_1 = [1, \infty)$ ). We recall that  $|\dot{\phi}_1|_{M_0} = \zeta(x, l)$ , where  $\zeta(x, l) \leq C_1 x^{-4}$  for  $x \in [a, \pi/2l]$  and  $\zeta(x, l) \leq C_1 (\pi/l - x)^{-4}$  for  $x \in [\pi/2l, b]$ .

Consider the equation

$$(l^{-2} \sin^2(lx)) Y'' - 2Y = 0 \quad (7)$$

with the boundary conditions that satisfy

$Y(l^{-1} \sin^{-1} l) = O(1)$  and  $Y(\pi/l - l^{-1} \sin^{-1} l) = O(1)$ , as  $l \rightarrow 0$ .

We claim that there exists some  $h = O(l)$  such that  $\ddot{\mathcal{H}} - h$  is a supersolution to (7) for  $x \in [l^{-1/4}, b - l^{-1/4}]$ . To see this, we notice that  $2|\dot{\phi}_0|^2|_{M_1} l^{-4} \sin^4(lx)$  decays rapidly as  $x \rightarrow \pi/2l$  for small  $l$ . So there is a positive number  $h = O(l)$  such that  $2|\dot{\phi}_0|^2|_{M_1} l^{-4} \sin^4(lx) < 2h$  for  $x \in [l^{-1/4}, b - l^{-1/4}]$ . Now for  $x \in [l^{-1/4}, b]$  we have

$$(l^{-2} \sin^2(lx))(\ddot{\mathcal{H}} - h)'' - 2(\ddot{\mathcal{H}} - h) = 2h - 2|\dot{\phi}_0|^2|_{M_1} l^{-4} \sin^4(lx) > 0 \quad (8)$$

Notice that for any constant  $\lambda$ , we have that if  $Y(x)$  solves the equation (7), then so does  $\lambda Y(x)$ . So up to multiplying by a bounded constant, we have  $Y|_{\partial M_1} > (\ddot{\mathcal{H}} - h)|_{\partial M_1}$ . Hence  $\ddot{\mathcal{H}} - h$  is a subsolution to (7) for  $x \in [l^{-1/4}, b - l^{-1/4}]$ . We can check the solutions to (7) have the form of

$$Y(l; x) = B_3 \cot(lx) + B_4(1 - lx \cot(lx))$$

where constants  $B_3 = B_3(l)$  and  $B_4 = B_4(l)$  satisfy, from the boundary conditions for the equation (7), that

$$B_3 = O(l) \text{ and } B_4 = O(l)$$

Therefore in  $[l^{-1/4}, b - l^{-1/4}]$ , We have  $\ddot{\mathcal{H}} \leq h + Y(x)$ . Now,

$$\begin{aligned} \int_{M_1} Y(x) |\mu_1|^2 \sigma dx dy &= \int_0^1 \int_a^b Y(x) (l^{-2} \sin^2(lx)) dx dy \\ &= \int_0^1 \int_a^b B_3 \cot(lx) (l^{-2} \sin^2(lx)) dx dy \\ &+ \int_0^1 \int_a^b B_4 (1 - lx \cot(lx)) (l^{-2} \sin^2(lx)) dx dy \\ &= O(l^{-2}) + O(l^{-2}) \\ &= O(l^{-2}) \end{aligned}$$

Also for  $x \in [a, l^{-1/4}] \cup [b - l^{-1/4}, b]$ , we apply maximum principle to equation (6) and find that

$$\ddot{\mathcal{H}} \leq \max(l^{-4} \sin^4(lx_0) \zeta^2(x_0, l), \sup(\ddot{\mathcal{H}}|_{\partial(( [a, l^{-1/4}] \cup [b - l^{-1/4}, b]) \times [0, 1])}))$$

for some  $x_0 \in [a, l^{-1/4}]$ . Hence using the properties of  $\zeta(x, l)$  in  $[a, b]$ , a direct computation shows that

$$\begin{aligned} \int_0^1 \int_a^{l^{-1/4}} \ddot{\mathcal{H}}(x) |\mu_1|^2 dA &= O(l^{-3/4}) = o(l^{-1}) \\ \int_0^1 \int_{b-l^{-1/4}}^b \ddot{\mathcal{H}}(x) |\mu_1|^2 dA &= O(l^{-3/4}) = o(l^{-1}) \end{aligned}$$

With this and using  $\ddot{\mathcal{H}} \leq h + Y(x)$  for  $x \in [l^{-1/4}, b - l^{-1/4}]$ , we have

$$\begin{aligned}
\int_{M_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA &= \int_0^1 \int_a^b \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 dA \\
&= \int_0^1 \int_{l^{-1/4}}^{b-l^{-1/4}} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 dA + \int_0^1 \int_a^{l^{-1/4}} \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 dA \\
&+ \int_0^1 \int_{b-l^{-1/4}}^b \ddot{\mathcal{H}}(x)|\dot{\mu}_1|^2 dA \\
&\leq \int_0^1 \int_a^b (Y(x) + h)|\dot{\mu}_1|^2 \sigma dx dy + o(l^{-1}) \\
&= O(l^{-2}) + O(l^{-2}) + o(l^{-1}) = O(l^{-2}). \tag{9}
\end{aligned}$$

Now combine the estimates of Lemma 7, (9), (10), we will have

$$\begin{aligned}
|R| &\leq 4 \int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy \\
&= 4 \left( \int_{M_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA + \int_{M_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA \right) \\
&= O(l^{-2}) \tag{10}
\end{aligned}$$

Now with Lemma 6 we have

$$|R|/\Pi = O(l^{-2})l^3 = O(l)$$

namely,  $|R|/\Pi$  tends to zero as we pinch the geodesics.

### 3.4 Constructed Maps

In last subsection, we estimated the terms in the curvature formula in the model case, i.e., the surface is a pair of cylinders. Essentially, in each pinching neighborhood  $M_i$ , we used the cylinder map  $w_i$  instead of the harmonic map  $W_i$  during the computation, for  $i = 0, 1$ . Now we are in the general setting, i.e., the surface  $\Sigma$  develops two nodes. In this subsection, we will construct a family of maps  $G_i$  to approximate the harmonic map  $W_i$ , and the essential parts of this family are the identity map of the surface restricted in the non-cylindrical part and the cylinder map in each pinching neighborhood. We will also show that this constructed family  $G_i$  is reasonably close to the harmonic maps  $W_i$ , for  $i = 0, 1$ ; hence we can use the estimates we obtained in the previous subsection to the general situation.

We recall some of the notations from previous subsections. We still set  $M_0$  and  $M_1$  to be the pinching neighborhoods of the nodes  $p_0$  and  $p_1$ , respectively. Also  $W_0(t)$  is the harmonic map corresponding to fixing  $\gamma_1$  in  $M_1$  having length  $l$ , and pinching  $\gamma_0$  in  $M_0$  into length  $L = L(t)$ , where  $L(0) = l$ . Similarly  $W_1(t)$  is the harmonic map corresponding to fixing  $\gamma_0$  in  $M_0$  into length  $l$ , and pinching  $\gamma_1$  in  $M_1$  having length  $L = L(t)$ . Let  $w_0(t)$  and  $w_1(t)$  be cylinder maps in the model case, we want to show that  $W_0(t)$  is close to  $w_0(t)$  in  $M_0$  and is close to identity map outside of  $M_0$ .

We denote subsets  $\Sigma_0 = \{p \in \Sigma : \text{dist}(p, \partial M_0) > 1\}$ , and  $\Sigma_1 = \{p \in \Sigma : \text{dist}(p, \partial M_1) > 1\}$ . Define the 1-tube of  $\partial M_0$  as  $B(\partial M_0, 1) = \{p \in \Sigma : \text{dist}(p, \partial M_0) \leq 1\}$ , and the 1-tube of  $\partial M_1$  as  $B(\partial M_1, 1) = \{p \in \Sigma : \text{dist}(p, \partial M_1) \leq 1\}$ . We can construct a  $C^{2,\alpha}$  map  $G_0 : \Sigma \rightarrow \Sigma$  such that

$$G_0(p) = \begin{cases} w_0(t)(p), & p \in M_0 \cap \Sigma_0 \\ p, & p \in (\Sigma_0 \setminus M_0) \\ g_t(p), & p \in B(\partial M_0, 1) \end{cases}$$

where  $g_t(p)$  in  $B(\partial M_0, 1)$  is constructed so that it satisfies:

- $g_t(p) = p$  for  $p \in \partial(\Sigma_0 \setminus (M_0 \cup B(\partial M_0, 1)))$ , and  $g_t(p) = w_0(t)(p)$  for  $p \in \partial(M_0 \cap \Sigma_0)$ ;
- $g_t$  is the identity map when  $t = 0$ ;
- $g_t$  is smooth and the tension of  $g_t$  is of the order  $O(t)$ .

We note that  $G_0$  consists of three parts. It is the cylinder map of  $M_0$  in  $M_0 \cap \Sigma_0$ , the identity map in  $\Sigma_0 \setminus M_0$ , and a smooth map in the intersection region  $B(\partial M_0, 1)$ . Among three parts of the constructed map  $G_0(t)$ , two of them, the identity map and the cylinder map, are harmonic, hence have zero tension; thus the tension of  $G_0(t)$  is concentrated in  $B(\partial M_0, 1)$ . From §3.2, for the cylinder map  $w_0 = u(x) + iy$ , we have  $u' = \sqrt{1 + c_0(t)L^{-2}\sin^2 Lu}$ , where  $c_0(0) = 0, \dot{c}_0(0) = 4$ . Hence for  $x \in [l^{-1}\sin^{-1}(l), l^{-1}\sin^{-1}(l) + 1]$ ,

$$\begin{aligned} w_{0,z}(x, y) &= \frac{1}{2}(u'(x) + 1) = \frac{1}{2}((1 + O(1)c_0(t))^{\frac{1}{2}} + 1) = 1 + O(1)t + O(t^2) \\ |w_{0,z}(x, y) - 1| &= O(t) \rightarrow 0, (t \rightarrow 0) \\ |w_{0,z\bar{z}}(x, y)| &= |\frac{1}{4}u''(x)| = O(|Lc_0(t)Lu(u'^2 - 1)|) = O(t) \end{aligned}$$

Thus we can require that  $|g_{t,z}| \leq C_2 t$  and  $|g_{t,z\bar{z}}| \leq C_2 t$ , and the constant  $C_2 = C_2(t, l)$  is bounded in both  $t$  and  $l$  since the coefficient of  $t$  for  $c_0(t)$  is bounded for small  $t$  and small  $l$ . With the local formula of the tension in section 2, we have  $\tau(G_0(t))$ , the tension of  $G_0(t)$  is of the order  $O(t)$ .

Similarly, we have a  $C^{2,\alpha}$  map  $G_1 : \Sigma \rightarrow \Sigma$ . Note that these constructed maps  $G_0$  and  $G_1$  are not necessarily harmonic.

Now we are about to compare the constructed family  $G_0(t)$  and the family of the harmonic maps  $W_0(t)$ . To do this, we consider the function  $Q_0 = \cosh(\text{dist}(W_0, G_0)) - 1$ .

**Lemma 9.**  *$\text{dist}(W_0, G_0) \leq C_3 t$  in  $B(\partial M_0, 1)$ , where the constant  $C_3 = C_3(t, l)$  is bounded for small  $t$  and  $l$ .*

First, we want to show that  $Q_0$  is a  $C^2$  function. Notice that both the harmonic map  $W_0(t)$  and the constructed map  $G_0(t)$  are the identity map when  $t = 0$ , and both families vary smoothly without changing homotopy type in  $t$  for sufficiently small  $|t|$  ([8]). For all  $l > 0$ , and for any  $\varepsilon > 0$ , there exists a  $\delta$  such that for  $|t| < \delta$ , we have  $|W_0(t) - W_0(0)| < \frac{\varepsilon}{2}$  and  $|G_0(t) - G_0(0)| < \frac{\varepsilon}{2}$ . Therefore the triangular inequality implies that  $|W_0(t) - G_0(t)| < \varepsilon$ . Since  $l$  is positive, the Collar Theorem ([6]) implies that the surface has positive injectivity radius  $r$  bounded below, and we choose our  $\varepsilon \ll r$ , so  $Q_0$  is well defined and smooth.

We follow an argument in [12]. For any unit  $v \in T^1(B(\partial M_0, 1))$ , the map  $G_0$  satisfies the inequality  $||dG_0(v)|| - 1| = O(t)$ , hence  $|dG_0(v)|^2 > 1 - \varepsilon_0$  where  $\varepsilon_0 = O(t)$ , then we find that for any  $x \in \Sigma$ ,

$$\begin{aligned} \Delta Q_0 &\geq \min\{|dG_0(v)|^2 : dG_0(v) \perp \gamma_x\} Q \\ &- < \tau(G_0), \nabla d(\bullet, W_0)|_{G_0(x)} > \sinh(\text{dist}(W_0, G_0)) \end{aligned} \quad (11)$$

where  $\gamma_x$  is the geodesic joining  $G_0(x)$  to  $W_0(x)$  with initial tangent vector  $-\nabla d(\bullet, W_0)|_{G_0(x)}$  and terminal tangent vector  $\nabla d(G_0(x), \bullet)|_{W_0(x)}$ . If  $G_0(t)$  does not coincide with  $W_0(t)$  on  $B(\partial M_0, 1)$ , we must have all maxima of  $Q_0(t)$  on the interior of  $B(\partial M_0, 1)$ , at any such maximum, we apply the inequality  $|dG_0(v)|^2 > 1 - \varepsilon_0$  to (12) to find

$$0 \geq \Delta Q_0 \geq (1 - \varepsilon_0) Q_0 - \tau(G_0)(\sinh(\text{dist}(W_0, G_0)))$$

so that at a maximum of  $Q_0$ , we have

$$Q_0 \leq \frac{\tau(G_0) \sinh \text{dist}(W_0, G_0)}{(1 - \varepsilon_0)}$$

We notice that  $Q_0$  is of the order  $\text{dist}^2(W_0, G_0)$  and  $\sinh \text{dist}(W_0, G_0)$  is of the order  $\text{dist}(W_0, G_0)$ , this implies that  $\text{dist}(W_0, G_0)$  is of the order  $O(t)$  in  $B(\partial M_0, 1)$ , which completes the proof of Lemma 8.

**Remark 10.** *Lemma 8 implies that  $Q_0(t)$  is of the order  $O(t^2)$  in  $B(\partial M_0, 1)$ .*

Note that  $B(\partial M_0, 1)$  contains the boundary of the cylinder  $M_0 \cap \Sigma_0$ , which we identify with  $[a + 1, b - 1] \times [0, 1]$ , where, again,  $a = a(l) = l^{-1} \sin^{-1}(l)$ , and  $b = b(l) = \pi l^{-1} - l^{-1} \sin^{-1}(l)$ . While in the cylinder  $M_0 \cap \Sigma_0$ , we have the inequality

$$\begin{aligned}
\Delta Q_0 &\geq (1 - \varepsilon_0)Q_0 - \tau(G_0)(\sinh(\text{dist}(W_0, G_0))) \\
&= (1 - \varepsilon_0)Q_0 - \tau(G_0)(\tanh(\text{dist}(W_0, G_0)))(1 + Q_0) \\
&= (1 - \varepsilon_0 - \tau(G_0))Q_0 - \tau(G_0)(\tanh(\text{dist}(W_0, G_0))) \\
&\geq 1/2Q_0 - C_4 t^2
\end{aligned}$$

where the constant  $C_4$  is bounded for small  $t$  and  $l$ . Therefore we find that  $Q_0(z, t)$  decays rapidly in  $z = (x, y)$  for  $x$  close enough to  $\pi/2l$ . Hence we can assume that  $\text{dist}(W_0, G_0)$  is at most of order  $C't$  in  $[a + 1, b - 1]$ , here  $C' = C'(x, l)$  is no greater than  $C_5 x^{-2}$  for  $x \in [a + 1, \pi/2l]$ , and no greater than  $C_5(\pi/l - x)^{-2}$  for  $x \in [\pi/2l, b - 1]$ , where  $C_5$  is bounded for small  $t$  and  $l$ . Both maps  $W_0$  and  $G_0$  are harmonic in  $M_0 \cap \Sigma_0$ , so they are also  $C^1$  close ([8]), i.e. we have  $|W_{0,\bar{z}} - G_{0,\bar{z}}| \leq C_5 x^{-2} t$  for small  $t$  and  $l$ , when  $x \in [a + 1, \pi/2l]$ . Thus we see that  $|\dot{W}_{0,\bar{z}} - \dot{G}_{0,\bar{z}}| = C_5 x^{-2}$ , for  $x \in [a + 1, \pi/2l]$ . Also,  $|\dot{W}_{0,\bar{z}} - \dot{G}_{0,\bar{z}}| = C_5(\pi/l - x)^{-2}$ , for  $x \in [\pi/2l, b - 1]$ .

As before we denote  $\phi_0$  and  $\phi_1$  as Hopf differentials corresponding to harmonic maps  $W_0(t)$  and  $W_1(t)$ , respectively. We also denote  $\mu_0$  and  $\mu_1$  as the corresponding Beltrami differentials in  $M_0 \cap \Sigma_0$  and  $M_1 \cap \Sigma_1$ , respectively. Write  $\phi_{G_0} = \rho G_{0,z}(t) \bar{G}_{0,z}(t)$  and  $\phi_{G_1} = \rho G_{1,z}(t) \bar{G}_{1,z}(t)$ . Notice that in  $M_0 \cap \Sigma_0$ , map  $G_0$  is the cylinder map hence harmonic, so  $\phi_{G_0}$  is the Hopf differential corresponding to  $G_0$ . When  $t = 0$  we have  $W_0 = G_0 = \text{identity}$  and  $\rho = \sigma$ , hence we can differentiate  $\phi_{G_0} = \rho G_{0,z}(t) \bar{G}_{0,z}(t)$  in  $t$  at  $t = 0$ , and find that  $|\dot{\phi}_{G_0} - \dot{\phi}_{W_0}| = \sigma |\dot{W}_{0,\bar{z}} - \dot{G}_{0,\bar{z}}| \leq C_5 l^2 x^{-2} \text{csc}^2(lx) = O(1)$  for  $x \in [a + 1, \pi/2l]$ , and  $|\dot{\phi}_{G_0} - \dot{\phi}_{W_0}| \leq C_5 l^2 (\pi/l - x)^{-2} \text{csc}^2(lx) = O(1)$  for  $x \in [\pi/2l, b - 1]$ . Therefore we have proved

**Lemma 11.**  $|\dot{\phi}_{G_0} - \dot{\phi}_{W_0}| = O(1)$  for  $x \in [a + 1, b - 1]$ .

Similarly we have  $|\dot{\phi}_0 - \dot{\phi}_{G_0}|_{M_1 \cap \Sigma_1} = O(1)$ , and  $|\dot{\phi}_1 - \dot{\phi}_{G_1}|_{M_0 \cap \Sigma_0} = O(1)$ , also  $|\dot{\phi}_1 - \dot{\phi}_{G_1}|_{M_1 \cap \Sigma_1} = O(1)$ .

### 3.5 Proof of the Main theorem

In last subsection, we constructed families of maps and found that they are reasonably close to the corresponding families of harmonic maps between surfaces. Now we are ready to adapt the estimates in the model case to the

general setting, and prove the proposition in this situation, which will imply our main theorem.

Recall from §3.3 that the sectional curvature is represented by  $R/\Pi$ . With Lemma 6 in the model case, the inequality  $\Pi \geq \Pi_{M_0 \cap \Sigma_0} \sim l^{-3}$  still holds. Hence  $|R|/\Pi = O(l^3)|R|$ , so it will be sufficient to show that  $|R| = o(l^{-3})$ . From Lemma 7, we have  $|R| \leq 4 \int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy$ , so we will show that Lemma 8 still holds, which implies desired curvature estimate immediately.

Now we are about to estimate  $\int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy$ , which breaks into 3 integrals as follows:

$$\begin{aligned} \int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy &= \int_{M_0 \cap \Sigma_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy \\ &+ \int_{M_1 \cap \Sigma_1} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy \\ &+ \int_K D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy \end{aligned} \quad (12)$$

where  $K$  is the compact set disjoint from  $(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)$ .

For the third integral, from the previous discussion, because of the convergence of the harmonic maps to the harmonic maps of “noded” problem, we have both  $|\dot{\mu}_0|$  and  $|\dot{\mu}_1|$  are bounded. The maximum principle implies that  $D(|\dot{\mu}_0|^2) = \mathcal{H} \leq \sup\{|\dot{\mu}_0|^2\}$ . Note that  $K$  is compact, hence we have the third integral is the order of  $O(1)$ .

From Lemma 4, we have  $(\Delta - 2)\ddot{\mathcal{H}} = -2\frac{|\dot{\phi}_0|^2}{\sigma^2}$ , so we can rewrite (12) as

$$\begin{aligned} \int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy &= \int_{M_0 \cap \Sigma_0} \ddot{\mathcal{H}}|\dot{\mu}_1|^2 \sigma dx dy \\ &+ \int_{M_1 \cap \Sigma_1} \ddot{\mathcal{H}}|\dot{\mu}_1|^2 \sigma dx dy + O(1) \end{aligned} \quad (13)$$

Now we will look at the first and the second integrals in (13). Recall that  $(\Delta - 2)\ddot{\mathcal{H}}^G = -2\frac{|\dot{\phi}_0^G|^2}{\sigma^2}$ , where  $\mathcal{H}^G$  is the holomorphic energy corresponding to the model case when the harmonic map is the cylinder map, and  $\phi_0^G$  is the quadratic differential corresponding to the constructed map  $G_0$ . We also denote  $\mu_0^G$  to be the Beltrami differential corresponding to  $\phi_0^G$ . We assign similar meanings for  $\phi_1^G$  and  $\mu_1^G$ . Also recall that  $|\dot{\phi}_0 - \dot{\phi}_0^G| = O(1)$  and  $|\dot{\phi}_1 - \dot{\phi}_1^G| = O(1)$  in  $(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)$ , here  $O(1)$  is bounded in  $l$  for small  $l$ . So we can set some  $\lambda = O(1)$  (bounded in  $l$  for small  $l$ ) such that  $|\dot{\phi}_0|^2 < \lambda^2 |\dot{\phi}_0^G|^2$  and at the boundary of  $(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)$

satisfies  $\ddot{\mathcal{H}} < \lambda \ddot{\mathcal{H}}^G$ . For example, we can take  $\lambda = 1 + \max_{\partial K}(\frac{\ddot{\mathcal{H}}}{\ddot{\mathcal{H}}^G}, \frac{|\dot{\phi}_0|}{|\dot{\phi}_0^G|})$ , this  $\lambda = O(1)$  because at  $\partial K = \partial((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1))$ , both  $\ddot{\mathcal{H}}$ ,  $\ddot{\mathcal{H}}^G$ , and  $\frac{|\dot{\phi}_0|}{|\dot{\phi}_0^G|}$  are bounded. Therefore,

$$(\Delta - 2)(\ddot{\mathcal{H}} - \lambda \ddot{\mathcal{H}}^G) = 2 \frac{\lambda^2 |\dot{\phi}_0^G|^2 - |\dot{\phi}_0|^2}{\sigma^2} > 0$$

So  $(\ddot{\mathcal{H}} - \lambda \ddot{\mathcal{H}}^G)$  is a subsolution to the equation  $(\Delta - 2)Y = 0$  whose solutions have the form of  $Y(l; x) = B_3 \cot(lx) + B_4(1 - lx \cot(lx))$ , where constants  $B_3$  and  $B_4$  satisfy that  $B_3 = O(l)$  and  $B_4 = O(l)$ . Hence in  $(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)$ , we have  $\ddot{\mathcal{H}} \leq \lambda \ddot{\mathcal{H}}^G + B_3 \cot(lx) + B_4(1 - lx \cot(lx))$ . We apply this into (13) and find that,

$$\begin{aligned} \int_{\Sigma} D(|\dot{\mu}_0|^2) |\dot{\mu}_1|^2 \sigma dx dy &= \int_{(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)} \ddot{\mathcal{H}} |\dot{\mu}_1|^2 \sigma dx dy + O(1) \\ &\leq \int_{(M_0 \cap \Sigma_0)} (\lambda \ddot{\mathcal{H}}^G + Y(l; x)) (|\dot{\mu}_1|^2) \sigma dx dy \\ &\quad + \int_{(M_1 \cap \Sigma_1)} (\lambda \ddot{\mathcal{H}}^G + Y(l; x)) (|\dot{\mu}_1|^2) \sigma dx dy + O(1) \\ &\leq \int_{M_0 \cap \Sigma_0} (\lambda \ddot{\mathcal{H}}^G + Y(l; x)) (2|\dot{\mu}_1|^2 + 2|\dot{\mu}_1 - \dot{\mu}_1^G|^2) dA \\ &\quad + \int_{M_1 \cap \Sigma_1} (\lambda \ddot{\mathcal{H}}^G + Y(l; x)) (2|\dot{\mu}_1|^2 + 2|\dot{\mu}_1 - \dot{\mu}_1^G|^2) dA \\ &\quad + O(1) \end{aligned}$$

Recalling the computation in §3.3, and  $|\dot{\mu}_1 - \dot{\mu}_1^G| = |\dot{\phi}_1 - \dot{\phi}_1^G|/\sigma$ , where  $|\dot{\phi}_1 - \dot{\phi}_1^G| \leq C_5 l^2 x^{-2} \csc^2(lx)$  for  $x \in [a+1, \pi/2l]$ , we have the following:

$$\begin{aligned} \int_{M_0} (\lambda \ddot{\mathcal{H}}^G) |\dot{\mu}_1^G|^2 \sigma dx dy &= O(l^{-2}) \\ \int_{M_1} (\lambda \ddot{\mathcal{H}}^G) |\dot{\mu}_1^G|^2 \sigma dx dy &= O(l^{-2}) \\ \int_{M_0} Y(l, x) |\dot{\mu}_1^G|^2 \sigma dx dy &= O(1) \\ \int_{M_1} Y(l, x) |\dot{\mu}_1^G|^2 \sigma dx dy &= O(l^{-2}) \\ \int_{M_0 \cap \Sigma_0} (\ddot{\mathcal{H}}^G + Y(l; x)) (|\dot{\mu}_1 - \dot{\mu}_1^G|^2) dA &= o(l^{-2}) \\ \int_{M_1 \cap \Sigma_1} (\ddot{\mathcal{H}}^G + Y(l; x)) (|\dot{\mu}_1 - \dot{\mu}_1^G|^2) dA &= o(l^{-2}) \end{aligned}$$

Therefore  $\int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy = O(l^{-2})$ , with Lemma 7, we have  $|R| \leq 4 \int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy = O(l^{-2})$ . Apply this into the curvature formula, we have  $|R|/\Pi = O(l) \rightarrow 0$  as  $l \rightarrow 0$ , which completes the proof of proposition.

## 4 Punctured Surface Case

In previous sections, the argument on a compact surface is enough to prove our main theorem. However, we remark here that we allow the surface has finitely many punctures.

For the case of punctured surfaces, the existence of a harmonic diffeomorphism between punctured surfaces has been investigated by Wolf ([26]) and Lohkamp ([14]). In particular, Lohkamp ([14]) showed that a homeomorphism between punctured surfaces is homotopic to a unique harmonic diffeomorphism with finite energy, and the holomorphic quadratic differential corresponding to the harmonic map in the homotopy class of the identity is a bijection between Teichmüller space of punctured surfaces and the space of holomorphic quadratic differentials.

In this case, the set  $\Sigma \setminus ((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1))$  is no longer compact. Let  $K_0$  be a compact surface with finitely many punctures, and  $\{K_m\}$  be a compact exhaustion of  $K_0$ . We now estimate  $\int_{K_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA$ . Let  $H(t)$  be the holomorphic energy corresponding to the harmonic map  $w(t) : (K_0, \sigma) \rightarrow (K_0, \rho(t))$ , then  $H(t)$  is bounded above and below, and has nodal limit 1 near the punctures ([26]), hence both  $|\dot{\mu}_0|^2$  and  $|\dot{\mu}_1|^2$  have the order  $o(1)$  near the punctures. To see this, we consider  $K_0$  as the union of  $K_m$  and disjoint union of finitely many punctured disks, each equipped hyperbolic metric  $\frac{|dz|^2}{z^2 \log^2 z}$ . Then  $|\dot{\mu}_0| = O(|z| \log^2 z) \rightarrow 0$  as  $z$  tends to the puncture, since the quadratic differential has a pole of at most the first order. A similar result holds for  $|\dot{\mu}_1|$ . We notice that  $\partial K_0$  is the boundary of the cylinders, where the harmonic maps converge to a solution to the ‘‘noded’’ problem as  $l \rightarrow 0$ , hence  $D(|\dot{\mu}_0|^2) = \ddot{H}(t)$  is bounded on  $\partial K_0$ . Therefore we apply Omori-Yau maximum principle ([18] [31]) to  $(\Delta - 2)\ddot{H} = -2|\dot{\mu}_0|^2$  on  $K_0$  and obtain that  $\sup(D(|\dot{\mu}_0|^2)) \leq \max(\sup(|\dot{\mu}_0|^2), \max(D(|\dot{\mu}_0|^2))|_{\partial K_0}) = O(1)$ . Hence we have

$$\begin{aligned} \int_{K_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA &\leq \int_{K_0} \sup(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA \\ &\leq O(1)O(1)Vol(K_0) \\ &= O(1) \end{aligned}$$

In other words, our proof carries over to the punctured case, which completes the proof of our Main theorem.

## References

- [1] W. Abikoff, Degenerating families of Riemann surfaces, *Annals of Math.* **105** (1977) 29-44
- [2] L. Ahlfors, Some Remarks on Teichmüller's Space of Riemann surfaces, *Ann. Math.* **74**(1961)171-191
- [3] Al'ber, Spaces of mappings into a manifold with negative curvature, *Sov. Math. Dokl.* **9** (1967) 6-9
- [4] L. Bers, Spaces of degenerating Riemann surfaces in discontinuous groups and Riemann surfaces, *Annals of Math Studies* **79** Princeton University, Princeton, New Jersey 1974
- [5] J. Brock, B. Farb, Curvature and rank of Teichmüller space, *preprint* 2001
- [6] P. Buser, Geometry and spectra of compact Riemann surfaces, Birkhäuser, Boston 1992
- [7] T. Chu, The Weil-Petersson metric in moduli space, *Chinese J. Math.* **4** (1976) 29-51
- [8] J. Eells, L. Lemaire, Deformations of metrics and associated harmonic mappings, *Proc. Indian Acad. Sci.* **90** (1981) No.1, 33-45
- [9] J. Eells, J. Sampson, harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964) 109-160
- [10] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order, *2nd ed.* Springer, 1983
- [11] P. Hartman, On homotopic harmonic maps, *Can. J. Math.* **19** (1967) 673-687
- [12] R. Hardt, M. Wolf, Harmonic Extensions of quasiconformal maps to hyperbolic space, *Indiana Univ. Math. J.* **46** (1997) 155-163
- [13] J. Jost, Two dimensional geometric variational problems, Wiley-Interscience 1990

- [14] J. Lohkamp, Harmonic diffeomorphisms and Teichmüller theory, *Manuscripta Math.* **71** (1991) 339-360
- [15] H. Masur, The extension of the Weil-Petersson metric to the boundary of Teichmüller space, *Duke Math J.* **43** (1976) 623-635
- [16] D. Mumford, Stability of projective varieties, *L'Enseignement Math.* **23** (1977) 39-110
- [17] S. Nag, Complex analytic theory of Teichmüller space, Wiley, New York, 1988
- [18] H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan* **19** (1967) 205-241
- [19] H. Royden, Intrinsic metrics on Teichmüller space, *Proc. Int. Cong. Math.* **2** (1974) 217-221
- [20] J. Sampson, Some properties and applications of harmonic mappings, *Ann. Sci. Ecole Norm. Sup.* **4** (1978) 211-228
- [21] G. Schumacher, harmonic maps of the moduli space of compact Riemann surfaces, *Math. Ann.* **275** (1986) 455-466
- [22] R. Schoen, S. Yau, harmonic maps and the topology of stable hypersurfaces and manifolds of non-negative Ricci curvature, *Comment Math. Helv.* **51** (1976) 333-341
- [23] A. Tromba, On an energy function for the Weil-Petersson metric on Teichmüller space, *Man. Math.* **59** (1987) 249-260
- [24] A. Tromba, Teichmüller theory in Riemannian geometry, Birkhauser Basel (1992)
- [25] M. Wolf, Teichmüller theory of harmonic maps, *J. Diff. Geom.* **29** (1989) 449-479
- [26] M. Wolf, Infinite energy harmonic maps and degeneration on hyperbolic surfaces in moduli space *J. Diff. Geom.*, **33** (1991) 487-539
- [27] M. Wolf, S. Wolpert, Real analytic structures on the moduli space of curves, *Amer. J. Math.* **114** (1992) 1079-1102
- [28] S. Wolpert, Noncompleteness of the Weil-Petersson metric for Teichmüller space, *Pacific J. Math.* **61** (1975) 573-577

- [29] S. Wolpert, Chern forms and the Riemann tensor for the moduli space of curves, *Invent. Math.* **85**(1986) 119-145
- [30] S. Wolpert, Geometry of the Weil-Petersson completion of Teichmüller space, *preprint* 2002
- [31] S.-T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28**(1986) 119-145

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