STABILITY OF THE SURFACE AREA PRESERVING MEAN CURVATURE FLOW IN EUCLIDEAN SPACE

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Abstract. The surface area preserving mean curvature flow is a mean curvature type flow with a global forcing term to keep the hypersurface area fixed. By iteration techniques, we show that the surface area preserving mean curvature flow in Euclidean space exists for all time and converges exponentially to a round sphere, if initially the L^2-norm of the traceless second fundamental form is small (but the initial hypersurface is not necessarily convex).

1. Introduction

Let M^n be a smooth, embedded, closed (compact, no boundary) n-dimensional manifold in $\mathbb{R}^{n+1}$, and we evolve it by the surface area preserving mean curvature flow, that is,

$$\frac{\partial F}{\partial t} = (1 - hH)\,\nu, \qquad F(\cdot, 0) = F_0(\cdot)\,. \tag{1.1}$$

Here $F_0 : M^n \to \mathbb{R}^{n+1}$ is the initial embedding, and $H = H(x,t)$ is the mean curvature and $\nu = \nu(x,t)$ is the outward unit normal vector of $M_t = F(\cdot, t)$ at point (x,t) (for simplicity, we simply write $(x,t) \in M_t$). And the function h is given by

$$h = h(t) = \frac{\int_{M_t} H\, d\mu}{\int_{M_t} H^2\, d\mu}\,, \tag{1.2}$$

where $d\mu = d\mu_t$ denotes the surface area element of the evolving surface M_t with respect to the induced metric $g(t)$. Clearly we have $H \not\equiv 0$ on M_0 since there is no closed minimal hypersurface in Euclidean space by the maximum principle (see e.g. [**CM11**]). A good monotonicity property of the surface area preserving mean curvature flow (1.1) is that the surface area of M_t remains unchanged and the volume of the $(n+1)$-dimensional region enclosed by M_t is non-decreasing along the flow, see Corollary 2.3. This flow is a normalized variant of the classical mean curvature flow which is the steepest descent flow for the area functional, c.f. the volume preserving mean curvature flow introduced by Huisken in [**Hui87**]. We shall point out the velocity of the surface area preserving mean curvature flow depends on a global term $1-hH$, and hence this flow is quite different from rescaling the mean curvature flow by dilation and reparametrization considered by Huisken [**Hui84**, §9] in his pioneer work on the mean curvature flow.

2010 *Mathematics Subject Classification.* Primary 53C44, Secondary 58J35.

We denote $A = \{a_{ij}\}$ as the second fundamental form of M_t and its traceless part as $\mathring{A} = A - \frac{H}{n}g$. Then we have $|\mathring{A}|^2 = |A|^2 - \frac{1}{n}H^2$. This quantity measures the roundness of the hypersurface.

In this paper, we prove the following theorem on the stability of this surface area preserving mean curvature flow:

Theorem 1.1. *Let $M_t^n \subset \mathbb{R}^{n+1}, n \geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow* (1.1) *for $t \in [0, T)$ with $T \leq \infty$. Assume that $h(0) > 0$. There exists $\epsilon > 0$, depending only on n, $h(0)$, the surface area of M_0, $\max_{M_0} |A|$ and the L^2-norms of the covariant derivatives of A on M_0, such that if*

$$\int_{M_0} |\mathring{A}|^2 \, d\mu \leq \epsilon \,, \tag{1.3}$$

then $T = \infty$ and the flow converges exponentially to a round sphere.

Remark 1.2. The general scheme of the proof is an iteration argument, and the idea of using this to prove dynamical stability of geometric flows seems to go back to ([**Ye93**]). The stability of the volume preserving mean curvature flow was studied by Escher-Simonett ([**ES98**]) and Li ([**Li09**]), under different sets of conditions. In [**McC03**], McCoy proved that the surface area preserving mean curvature flow exists for all time and converges to a sphere if the initial hypersurface is *strictly convex*. As in the case of volume preserving mean curvature flow initiated by Huisken in [**Hui87**], strict convexity of the initial surface is essential. In our setting, we do not assume such strict convexity (or mean convexity) for the initial hypersurface. While it is crucial to keep track of the behavior of the global term $h(t)$ along the flow, the analytical nature of our case, namely the surface area preserving mean curvature flow, is much more complicated than that of the volume preserving mean curvature flow, since the function $h(t)$ contains two integral terms both involving the mean curvature. A key reduction in §3.2 for our treatment is that we may assume the H of the hypersurface is small (possibly changing signs), otherwise, the hypersurface is strictly convex already. As a result, the flow exists for all time and we iterate to prove the convergence to a round sphere. Our approach is expected to use to investigate the more general mixed volume preserving mean curvature flow studied by McCoy in [**McC04**], also in the study of the dynamical stability for the mean curvature flow [**LS13**].

Outline of the proof: Our strategy is as follows: based on the initial bounds, we prove bounds on some time interval for several geometric quantities (Theorem 3.2), and these bounds together with Lemma 3.4 allow us to make a reduction on the argument such that we have control on the mean curvature over the time interval, then we prove decay for these quantities on the time interval (Theorem 4.1). Main theorem then follows. One of the key ingredients in the proof is a version of the classical Michael-Simon inequality to derive the exponential decay for $\int_{M_t} |\mathring{A}|^2$.

Plan of the paper: There are four sections. In §2, we collect evolution equations for various geometric quantities associated to this flow, and provide some classic results that will be used in the proof. The proof of the main theorem is contained in the last two sections: we provide key estimates for the initial time interval of the iteration in §3, and we prove decay for $\int_{M_t} |\mathring{A}|^2$ and other quantities, and prove the Lemma 3.4 to use it later for a reduction of the argument, we then use these estimates to prove the long-time existence and convergence in §4.

Acknowledgements. Z. H. acknowledges support from U.S. national science foundation grants DMS 1107452, 1107263, 1107367 "RNMS: Geometric Structures and Representation varieties" (the GEAR Network), and he is also partially supported by a PSC-CUNY research award.

2. Preliminaries

For convenience of the reader, we collect some necessary preliminary results in this section. In §2.1, we obtain evolution equations for some key quantities and operators, many of which were derived in [**McC03**]; in §2.2, we state and use Hamilton's interpolation inequalities for tensors to obtain a L^2 estimate (Lemma 2.11) on the covariant derivatives of the tensor $\mathring{A}$. A version of the parabolic maximum principle and a version of the Michael-Simon inequality are also provided in this subsection.

2.1. Evolution of geometric quantities.

We start with the short time existence of the surface area preserving mean curvature flow (1.1) that is guaranteed by a work of Pihan:

Theorem 2.1. *([**Pih98**]) Let M_0 be a smooth embedded compact n-dimensional manifold in $\mathbb{R}^{n+1}$. Assume that $H \neq 0$ at some point of M_0 and $h(0) > 0$, then there exists $T_0 > 0$ such that the surface area preserving mean curvature flow (1.1) exists and is smooth for $t \in [0, T_0)$.*

We now collect and derive some evolution equations of several geometric quantities which will be used later. These quantities are:

(1) the induced metric of the evolving surface M_t: $g(t) = \{g_{ij}(t)\}$;
(2) the second fundamental form of M_t: $A(\bullet, t) = \{a_{ij}(\bullet, t)\}$, and its square norm given by
$$|A(\bullet, t)|^2 = g^{ij} g^{kl} a_{ik} a_{jl};$$
(3) the mean curvature of M_t with respect to the outward normal vector: $H(\bullet, t) = g^{ij} a_{ij}$;
(4) the traceless part of the second fundamental form: $\mathring{A} = A - \frac{H}{n} g$;
(5) the surface area element of M_t: $d\mu_t = \sqrt{det(g_{ij})}$.

Lemma 2.2. *([**McC03**]) The metric of M_t satisfies the evolution equation*

$$\frac{\partial}{\partial t} g_{ij} = 2(1 - hH) a_{ij} \,. \tag{2.1}$$

Therefore,

$$\frac{\partial}{\partial t} g^{ij} = -2(1 - hH) a^{ij} \tag{2.2}$$

and

$$\frac{\partial}{\partial t}(d\mu_t) = H(1 - hH)d\mu_t. \tag{2.3}$$

Moreover, the outward unit normal ν to M_t satisfies

$$\frac{\partial \nu}{\partial t} = h\nabla H\,. \tag{2.4}$$

As an easy consequence of (2.3), we have

Corollary 2.3. *([McC03])*

(1) *The surface area $|M_t|$ of M_t remains unchanged along the flow, i.e.,*
$$\frac{d}{dt}\int_{M_t} d\mu = \int_{M_t}(1 - hH)H\, d\mu = 0\,.$$

(2) *The volume of E_t, the $(n+1)$-dimensional region enclosed by M_t, is non-decreasing along the flow, i.e.,*
$$\frac{d}{dt}Vol(E_t) = \int_{M_t} d\mu - \frac{\left(\int_{M_t} H\, d\mu\right)^2}{\int_{M_t} H^2\, d\mu} \geq 0\,.$$

Remark 2.4. In Euclidean space, among all closed hypersurfaces, the sphere is of the least surface area with fixed enclosed volume, and as well as of the largest enclosed volume with fixed surface area. Therefore from this point of view, it is natural to study the sphere via both the volume preserving mean curvature flow and the surface area preserving mean curvature flow.

Theorem 2.5. *([McC03]) The second fundamental form satisfies the following evolution equation:*

$$\frac{\partial}{\partial t}a_{ij} = h\Delta a_{ij} + (1 - 2hH)a_i^m a_{mj} + h|A|^2 a_{ij}\,, \tag{2.5}$$

where $a_i^m = g^{ml}a_{li}$.

Corollary 2.6. *([McC03]) We have the evolution equations for H, $|A|^2$ and $|\mathring{A}|^2$:*

(i) $\frac{\partial}{\partial t}H = h\Delta H - (1 - hH)|A|^2$;
(ii) $\frac{\partial}{\partial t}|A|^2 = h\left(\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4\right) - 2tr\left(A^3\right)$,

where $tr\left(A^3\right) = g^{ij}g^{kl}g^{mn}a_{ik}a_{lm}a_{nj}$. Therefore we also have

(iii) $\frac{\partial}{\partial t}|\mathring{A}|^2 = h\Delta|\mathring{A}|^2 - 2h|\nabla\mathring{A}|^2 + 2h|A|^2|\mathring{A}|^2 - 2\left(tr(\mathring{A}^3) + \frac{2}{n}H|\mathring{A}|^2\right)$, *where* $|\nabla\mathring{A}|^2 = |\nabla A|^2 - \frac{1}{n}|\nabla H|^2$.

Proof. The last equation here is equivalent to the one from [McC03]. To see this, we used the following fact (see page 335 of [Li09]):

$$\text{tr}\left(A^3\right) - \frac{1}{n}|A|^2 H = \text{tr}\left(\mathring{A}^3\right) + \frac{2}{n}|\mathring{A}|^2 H.$$

□

We can then derive the evolution equations for the square norm of the covariant derivatives of the second fundamental form.

Corollary 2.7. *We have the evolution equation for* $|\nabla^m A|^2$*:*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 =& h\Delta|\nabla^m A|^2 - 2h|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A *_h \nabla^j A * \nabla^k A * \nabla^m A \\ &+ \sum_{r+s=m} \nabla^r A *_h \nabla^s A * \nabla^m A\,,\end{aligned} \tag{2.6}$$

where $*_h$ *and* $*$ *denote any linear combination of tensors formed by contraction by the metric* g *(*$*_h$ *means the coefficient contains a linear factor* h*).*

Proof. The time derivative of the Christoffel symbols Γ^i_{jk} is equal to

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\ &= g^{il}\left\{\nabla_j\left((1-hH)a_{kl}\right) + \nabla_k\left((1-hH)a_{jl}\right) - \nabla_l\left((1-hH)a_{jk}\right)\right\} \\ &= A *_h \nabla A + \nabla A\,,\end{aligned}$$

Here we have used the evolution equation for the metric, i.e., (2.1). Now we can proceed exactly as in [**Ham82**, §13] (see also [**Hui84**, §7]) to obtain (2.6). □

In addition, we prove the following lemma on the time-derivative of the function $h(t) = \frac{\int_{M_t} H\,d\mu}{\int_{M_t} H^2 d\mu}$. Later in §4.3, we will use it to establish a positive lower bound for $h(t)$ under our conditions.

Lemma 2.8.

$$\frac{dh}{dt} = \frac{\int_{M_t}[-(1-2hH)(1-hH)|A|^2 + H^2(1-hH)^2 + 2h^2|\nabla H|^2\,]d\mu}{\int_{M_t} H^2\,d\mu}.$$

Proof. For the sake of completeness, we compute as follows:

$$\begin{aligned}\frac{dh}{dt} &= \frac{d}{dt}\left(\frac{\int_{M_t} H\,d\mu}{\int_{M_t} H^2\,d\mu}\right) \\ &= \left(\int_{M_t} H^2\,d\mu\right)^{-1}\left[\int_{M_t} -(1-hH)|A|^2 + H^2(1-hH)\,d\mu\right] \\ &\quad - \left(\int_{M_t} H^2\,d\mu\right)^{-1}\left[\int_{M_t} -2h^2|\nabla H|^2 - 2hH(1-hH)|A|^2 + hH^3(1-hH)\,d\mu\right] \\ &= \frac{\int_{M_t}[-(1-2hH)(1-hH)|A|^2 + H^2(1-hH)^2 + 2h^2|\nabla H|^2\,]d\mu}{\int_{M_t} H^2\,d\mu}\,.\end{aligned}$$

□

2.2. **Interpolations, Michael-Simon's inequality and maximum principle.** We will also need to make use of several well-known techniques for our proof. We start with the following Hamilton's interpolation inequality for tensors.

Theorem 2.9. *([Ham82]) Let M^n be an n-dimensional compact Riemannian manifold and Ω be any tensor on M. Suppose*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r} \quad \text{with } r \geq 1\,.$$

We have the estimate

$$\left(\int_M |\nabla\Omega|^{2r}\,d\mu\right)^{1/r} \leq (2r-2+n)\left(\int_M |\nabla^2\Omega|^p\,d\mu\right)^{1/p}\left(\int_M |\Omega|^q\,d\mu\right)^{1/q}.$$

A consequence of this theorem is the following:

Corollary 2.10. *([Ham82]) Let M^n and Ω be the same as the Theorem 2.9. If $1 \leq i \leq m-1$, then there exists a constant $C = C(n,m)$ which is independent of the metric and connection on M, such that the following estimate holds:*

$$\int_M |\nabla^i\Omega|^{\frac{2m}{i}}\,d\mu \leq C\,\max_M |\Omega|^{2(\frac{m}{i}-1)}\int_M |\nabla^m\Omega|^2\,d\mu\,.$$

As an application of these interpolation inequalities, we prove an estimate that will be used later.

Lemma 2.11. *For any $m \geq 1$ we have the estimate*

$$\frac{d}{dt}\int_{M_t} |\nabla^m A|^2\,d\mu + 2h\int_{M_t} |\nabla^{m+1}A|^2\,d\mu$$
$$\leq C(n,m)\,(|h(t)|+1)\,\max_{M_t}\left(|A|^2+|A|\right)\int_{M_t} |\nabla^m A|^2\,d\mu\,.$$

Proof. By integrating the equation (2.6), and using the generalized Hölder inequality, for any $i,j,k,r,s > 0$ with $i+j+k = r+s = m$ we have

$$\frac{d}{dt}\int_{M_t} |\nabla^m A|^2\,d\mu - \int_{M_t}(1-hH)H|\nabla^m A|^2\,d\mu + 2h\int_{M_t} |\nabla^{m+1}A|^2\,d\mu$$
$$\leq C(n,m)(|h(t)|+1)\left(\int_{M_t} |\nabla^m A|^2\right)^{\frac{1}{2}}\left\{\left(\int_{M_t} |\nabla^r A|^{\frac{2m}{r}}\right)^{\frac{r}{2m}}\left(\int_{M_t} |\nabla^s A|^{\frac{2m}{s}}\right)^{\frac{s}{2m}}\right.$$
$$\left.+\left(\int_{M_t} |\nabla^i A|^{\frac{2m}{i}}\right)^{\frac{i}{2m}}\left(\int_{M_t} |\nabla^j A|^{\frac{2m}{j}}\right)^{\frac{j}{2m}}\left(\int_{M_t} |\nabla^k A|^{\frac{2m}{k}}\right)^{\frac{k}{2m}}\right\}.$$

The $|h|+1$ term comes from the fact that the contraction $*_h$ involves a linear factor h. We then apply Corollary 2.10 for tensor A to get

$$\left(\int_{M_t} |\nabla^q A|^{\frac{2m}{q}}\,d\mu\right) \leq C(n,m)\,\max_{M_t}|A|^{2(\frac{m}{q}-1)}\left(\int_{M_t} |\nabla^m A|^2\,d\mu\right),$$

where $q = i,j,k,r$ or s.

Also note that

$$\int_{M_t} |(1-hH)H|\nabla^m A|^2 d\mu \quad \leq \quad \max_{M_t}\{|H|+|h|H^2\}\int_{M_t} |\nabla^m A|^2 d\mu$$

$$\leq \quad C(n)(|h(t)|+1)\max_{M_t}\left(|A|^2+|A|\right)\int_{M_t}|\nabla^m A|^2 d\mu.$$

Now the conclusion follows from combining these inequalities. □

In order to carry out our proof of Theorem 4.1, we need the following version of Michael-Simon's inequality. A key point for applications in our setting is that this inequality is essentially a Poincaré type inequality for closed hypersurfaces which have small mean curvatures, see (3.22), (3.23), and (3.24).

Lemma 2.12. *Let M be a closed n-dimensional hypersurface, smoothly immersed in $\mathbb{R}^{n+1}$. Let $v \geq 0$ be any Lipschitz function on M. We have:*

(i) *For any $n > 2$,*

$$\left(\int_M v^{\frac{2n}{n-2}}\,d\mu\right)^{\frac{n-2}{n}} \leq C(n)\left(\int_M |\nabla v|^2\,d\mu + \int_M H^2 v^2\,d\mu\right). \tag{2.7}$$

(ii) *For $n = 2$,*

$$\int_M v^2 \leq C(n)\left(\int_M |\nabla v|^2\,d\mu + \int_M H^2 v^2\,d\mu\right). \tag{2.8}$$

Proof. See e.g. [**LS13**]. □

We will need the following version of the maximum principle, especially in the proof of Theorem 3.2.

Theorem 2.13. *(Maximum principle, see e.g.* [**CLN06**, Lemma 2.12]*) Suppose $u : M\times[0,T]\to\mathbb{R}$ satisfies*

$$\frac{\partial}{\partial t}u \leq a^{ij}(t)\nabla_i\nabla_j u + \langle B(t),\nabla u\rangle + F(u)\,,$$

where the coefficient matrix $\left(a^{ij}(t)\right) > 0$ for all $t\in[0,T]$, $B(t)$ is a time-dependent vector field and F is a Lipschitz function. If $u \leq c$ at $t=0$ for some $c>0$, then $u(x,t)\leq U(t)$ for all $(x,t)\in M_t, t\geq 0$, where $U(t)$ is the solution to the following initial value problem:

$$\frac{d}{dt}U(t) = F(U) \quad \textit{with} \quad U(0)=c\,.$$

3. Proof of Theorem 1.1: estimates and reduction

Our proof will occupy the rest of the paper, which is broken into two sections. In this section, we provide key estimates: the L^∞-bound for $|\mathring{A}|$ from its L^2-bound, and we make an important reduction before we proceed to complete the proof in next section.

3.1. **Establishing bounds for geometric quantities.** Let us start with a result of Topping which plays an important role in the key estimates in this subsection.

Lemma 3.1. *([**Top08**]) Let M be an n-dimensional closed, connected manifold smoothly immersed in $\mathbb{R}^N$, where $N \geq n+1$. Then the intrinsic diameter and the mean curvature H of M are related by*

$$diam(M) \leq C(n) \int_M |H|^{n-1} \, d\mu \, .$$

We now begin to prove the following key estimates which allows us to obtain the L^∞-bound for $|\mathring{A}|$ and $|\nabla H|$ on some time interval. More specifically,

Theorem 3.2. *Let $M_t^n \subset \mathbb{R}^{n+1}, n \geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow (1.1) for $t \in [0,T)$, with $T \leq \infty$. Assume that*

$$\max\left\{\max_{M_0}|A|, \int_{M_0} |\nabla^m A|^2 \, d\mu\right\} \leq \Lambda_0 \quad \textit{and} \quad h(0) \geq \frac{1}{\Lambda_1} \tag{3.1}$$

for some $\Lambda_0, \Lambda_1 > 0$ and all $m \in [1, \widehat{m}]$ with some $\widehat{m} \gg 1$. Then there exists some $\epsilon_0 = \epsilon_0(n, \Lambda_0, \Lambda_1, |M_0|) > 0$ and $T_1 = T_1(n, \Lambda_0, |M_0|) \in (0,1)$, such that if

$$\int_{M_0} |\mathring{A}|^2 \, d\mu \leq \epsilon_0 \, , \tag{3.2}$$

then either at some time $t_0 \in [0, T_1]$ the hypersurface becomes strictly convex and the flow converges exponentially to a round sphere as $t \to \infty$, or for all $t \in [0, T_1]$ we have

$$\max\left\{\max_{M_t}|A|, \int_{M_t} |\nabla^m A|^2 \, d\mu\right\} \leq 2\Lambda_0 \quad \textit{and} \quad h(t) \geq \frac{1}{2\Lambda_1}\, , \tag{3.3}$$

and moreover there exists $C_1 = C_1(n, \Lambda_0, |M_0|)$ and some universal constant $\alpha \in (0,1)$ such that for any $t \in [0, T_1]$

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H|\right) \leq C_1 \epsilon_0^\alpha \, . \tag{3.4}$$

Proof. The proof will begin here, but will be completed in next subsection. To begin, by the short time continuity, we denote $t_1 > 0$ as the maximal time such that for all $t \in [0, t_1]$ we have

$$\max\left\{\max_{M_t}|A|, \int_{M_t} |\nabla^m A|^2 \, d\mu\right\} \leq 2\Lambda_0 \quad \text{and} \quad h(t) \geq \frac{1}{2\Lambda_1}\, . \tag{3.5}$$

We also note that the following general inequality holds for the mean curvature H of any closed hypersurfaces M_t, namely,

$$\int_{M_t} |H|^n d\mu \geq \omega_n, \tag{3.6}$$

where ω_n is the area of the unit n-sphere, see e.g. [**Che71**].

By (3.5) and (3.6), and $|H| \leq \sqrt{n}|A| \leq 2\sqrt{n}\Lambda_0$, for any $t \in [0, t_1]$, we have

$$(2\sqrt{n}\Lambda_0)^{n-2} \int_{M_t} H^2 d\mu \geq \int_{M_t} |H|^2 |H|^{n-2} d\mu = \int_{M_t} |H|^n d\mu \geq \omega_n,$$

and so that we obtain the following lower bound for the integral $\int_{M_t} H^2$:

$$\int_{M_t} H^2 d\mu \geq \omega_n (2\sqrt{n}\Lambda_0)^{2-n}\,. \tag{3.7}$$

Therefore, since $|M_t| = |M_0|$ for any $t \in [0, t_1]$, we have

$$0 < h(t) = \frac{\int_{M_t} H\, d\mu}{\int_{M_t} H^2\, d\mu} \leq (\omega_n)^{-1}|M_0|(2\sqrt{n}\Lambda_0)^{n-1} := \Lambda_2(n, \Lambda_0, |M_0|)\,. \tag{3.8}$$

Now using the fact that $|\mathrm{tr}\left(A^3\right)| \leq |A|^3$ (see Lemma 2.2 of [**HS99**]), and Kato's inequality $|\nabla|A|| \leq |\nabla A|$, we derive from (ii) of Corollary 2.6 to find

$$\frac{\partial}{\partial t}|A| \leq h\Delta|A| + \Lambda_2|A|^3 + |A|^2 \quad \text{on } M_t \text{ for all } t \in [0, t_1]\,.$$

Then by the comparison maximum principle (Theorem 2.13), we have:

$$\max_{M_t}|A| \leq U(t) \text{ for all } t \in [0, t_1]\,, \text{ with } U(0) = \Lambda_0,$$

where $U(t) > 0$ solves

$$\Lambda_2 \ln\left(\Lambda_2 + \frac{1}{U}\right) - \frac{1}{U} = t + \Lambda_2 \ln\left(\Lambda_2 + \frac{1}{\Lambda_0}\right) - \frac{1}{\Lambda_0}\,.$$

Therefore, there exists $0 < t_2 = t_2(\Lambda_0, \Lambda_2) = t_2(n, \Lambda_0, |M_0|) \leq 1$ such that

$$\max_{M_t}|A| \leq \frac{3\Lambda_0}{2} \quad \text{for all } t \in [0, t_2]\,. \tag{3.9}$$

The first assertion of the Theorem, namely, (3.3), is obtained from the following technical lemma by setting $T_1 = \min\{t_1, t_2\}$.

Lemma 3.3. *There exists some constant $\epsilon = \epsilon(n, \Lambda_0, \Lambda_1, |M_0|) > 0$ such that if $\int_{M_0} |\mathring{A}|^2\, d\mu \leq \epsilon$, then*

$$t_1 \geq t_2 = t_2(n, \Lambda_0, |M_0|)\,.$$

Proof. (of the Lemma 3.3): Suppose this is not the case, then we have $t_1 < t_2 \in (0, 1]$. Then by the definition of t_1 (i.e., (3.5)) and the definition of t_2 (i.e., (3.9)), we conclude that at time $t = t_1$ there are two possibilities: (1) $\int_{M_t} |\nabla^m A|^2\, d\mu$ achieves the extreme value $2\Lambda_0$; or (2) $h(t)$ achieves the extreme value $\frac{1}{2\Lambda_1}$. Our strategy is to eliminate both possibilities.

We first integrate the evolution equation for $|\mathring{A}|^2$, namely, the equation (iii) of the Corollary 2.6 over M_t for $t \in [0, t_1]$, to obtain

$$\begin{aligned} &\frac{d}{dt}\int_{M_t} |\mathring{A}|^2\, d\mu - \int_{M_t} |\mathring{A}|^2 H(1 - hH)\, d\mu \\ &= \int_{M_t} \left[-2h|\nabla\mathring{A}|^2 + 2h|A|^2|\mathring{A}|^2 - 2\left(\mathrm{tr}(\mathring{A}^3) + \frac{2}{n}H|\mathring{A}|^2\right)\right] d\mu\,, \end{aligned} \tag{3.10}$$

and therefore by (3.5) and (3.8), we have

$$\frac{d}{dt}\int_{M_t} |\mathring{A}|^2\, d\mu \leq C(n, \Lambda_0, |M_0|)\int_{M_t} |\mathring{A}|^2 d\mu \quad \text{for all } t \in [0, t_1]\,, \tag{3.11}$$

where we have also used the following inequalities: $|H| \leq \sqrt{n}|A| \leq 2\sqrt{n}\Lambda_0$ and $|\mathrm{tr}(\mathring{A}^3)| \leq |\mathring{A}|^3 \leq 2\Lambda_0|\mathring{A}|^2$.

Therefore, using the inequality (3.11), we have

$$\int_{M_t} |\mathring{A}|^2 \, d\mu \leq \epsilon e^{C(n,\Lambda_0,|M_0|)t} \leq C_2(n,\Lambda_0,|M_0|)\epsilon \quad \text{for all } t \in [0,t_1]\,. \tag{3.12}$$

Now we apply Hamilton's interpolation inequality (Theorem 2.9, with $r = 1, p = q = 2$) to find for any $t \in [0,t_1]$:

$$\int_{M_t} |\nabla \mathring{A}|^2 \, d\mu \leq n \left(\int_{M_t} |\mathring{A}|^2 \, d\mu\right)^{\frac{1}{2}} \left(\int_{M_t} |\nabla^2 \mathring{A}|^2 \, d\mu\right)^{\frac{1}{2}} \leq C(n,\Lambda_0,|M_0|)\epsilon^{\frac{1}{2}}\,, \tag{3.13}$$

where we used $|\nabla^2 \mathring{A}| \leq C(n)|\nabla^2 A|$ and (3.5). In fact, using (3.5) and applying Theorem 2.9 inductively, we have, for all $m \in [1,\widehat{m}]$ and $t \in [0,t_1]$,

$$\int_{M_t} |\nabla^m \mathring{A}|^2 \, d\mu \leq C(n,\Lambda_0,|M_0|)\epsilon^{\frac{1}{2m}}\,. \tag{3.14}$$

This together with Corollary 2.10 imply that, for all $t \in [0,t_1]$,

$$\int_{M_t} |\nabla^m \mathring{A}|^p \, d\mu \leq C(n,\Lambda_0,|M_0|)\epsilon^{\beta}$$

for all $m \in [1,\widehat{m}]$ and any $p \leq \widehat{p}$ for some $\widehat{p}$ sufficiently large (note that the geometry of M_t is uniformly bounded, c.f. (3.5), so that the standard Sobolev embedding $C^\gamma \hookrightarrow W^{1,\widehat{p}}$ holds for some $\gamma > 0$, see e.g. [**Aub98**, §2]). Here $\beta > 0$ is some positive constants depending on n and the fixed constants $\widehat{m}$ and $\widehat{p}$. This yields, by the standard Sobolev inequality, that for some universal constant $\alpha \in (0,1)$ that is smaller than β, we have

$$\max_{M_t} |\nabla^m \mathring{A}| \leq C(n,\Lambda_0,|M_0|)\epsilon^{\alpha}. \tag{3.15}$$

for all $m \in [1,\widehat{m}]$ and $t \in [0,t_1]$. In particular, using [**Hui84**, Lemma 2.2], we have

$$\max_{M_t}(|\nabla H| + |\nabla \mathring{A}|) \leq C_3(n,\Lambda_0,|M_0|)\epsilon^{\alpha}\,, \tag{3.16}$$

for all $t \in [0,t_1]$.

One can then deduce from inequalities (3.12) and (3.16), and the fact that we always have $|M_t| = |M_0|$, to find

$$\begin{aligned} \max_{M_t} |\mathring{A}| &\leq \sqrt{\frac{C_2(n,\Lambda_0,|M_0|)\epsilon}{|M_0|}} + C_3(n,\Lambda_0,|M_0|)\epsilon^{\alpha} \cdot (\text{diameter of } M_t) \\ &\leq \sqrt{\frac{C_2(n,\Lambda_0,|M_0|)\epsilon}{|M_0|}} + C_4(n,\Lambda_0,|M_0|)\epsilon^{\alpha}\,. \end{aligned} \tag{3.17}$$

where $C_4(n,\Lambda_0,|M_0|) = C_3(n,\Lambda_0,|M_0|)|2\sqrt{n}\Lambda_0|^{n-1}|M_0|$ and in the last inequality we used Topping's Lemma 3.1 and $|H| \leq 2\sqrt{n}\Lambda_0$. We will continue the proof of this lemma after we establish some decaying properties of the quantities $\int_{M_t} |\nabla^m A|^2 \, d\mu$ and $h(t)$ in subsection 3.3, see Lemma 3.5 and Corollary 3.6.

3.2. **Reduction.** We want to differentiate the notions of being "sufficient smallness" and "arbitrary smallness" of the constant ϵ in the initial conditions. If ϵ is too small comparing with the mean curvature H, then the smallness of $|\mathring{A}|$ will force the initial hypersurface to be *strictly convex*, for which the classic results apply. The most interesting case occurs when the constant ϵ is within the appropriate range (depending on the initial bounds) and the initial hypersurface is allowed to be *non-convex*. There is no yet general results concerning the surface area preserving mean curvature flow starting from non-convex hypersurfaces in the literature. The behavior of global flows such as the surface area preserving mean curvature flow, starting from general (non-convex) hypersurface is expected to be very complicated and singularities are expected in finite time.

We want to make a key reduction in this subsection, which is that we may assume the mean curvature is not too big. Otherwise, with a bound on $|\nabla H|$, we can apply the following Lemma 3.4 to find that the hypersurface is already strictly convex. We now continue the proofs of Lemma 3.3 and Theorem 3.2. Note that $H = \sum_{i=1}^n \lambda_i$ and $|\mathring{A}| = \sqrt{\frac{1}{n}\sum_{i<j}(\lambda_i - \lambda_j)^2}$, then we deduce that every principal curvature

$$\lambda_i > 0 \quad \text{and } M_0 \text{ is } \textit{strictly convex} \tag{3.18}$$

if H and $|\mathring{A}|$ satisfy some inequality. In particular, we have

Lemma 3.4. *If a closed hypersurface satisfies that $|H| \geq n(n-1)|\mathring{A}| + \varepsilon$ for some $\varepsilon > 0$, then either $\lambda_i > 0$ for all i, or $\lambda_i < 0$ for all i.*

Proof. Let us only prove the case $H \geq n(n-1)|\mathring{A}| + \varepsilon$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be principal curvatures. Then we have

$$\sum_{i<j} |\lambda_i - \lambda_j| = H - n\lambda_1 + \sum_{1<i<j} |\lambda_i - \lambda_j|.$$

But we also find

$$\begin{aligned} \sum_{i<j} |\lambda_i - \lambda_j| &= \sum_{j=2}^n \sum_{i=1}^j (\lambda_j - \lambda_i) \\ &\leq \sum_{j=2}^n \sqrt{n \sum_{i=1}^j (\lambda_j - \lambda_i)^2} \\ &\leq n \sum_{j=2}^n |\mathring{A}| \\ &= n(n-1)|\mathring{A}|. \end{aligned}$$

Therefore we have

$$n\lambda_1 \geq H - n(n-1)|\mathring{A}| + \sum_{1<i<j} |\lambda_i - \lambda_j| \geq \varepsilon > 0.$$

$\square$

Once we reached (3.18), the classic results for strictly convex initial hypersurfaces apply, and the surface area preserving mean curvature flow exists for all time and converges to a round sphere, see [**McC03**]. With this in mind, we define the following constant:

(3.19)
$$\tilde{\epsilon} =: C_4(n,\Lambda_0,|M_0|)\epsilon^\alpha + 2n(n-1)\left(\sqrt{\frac{C_2(n,\Lambda_0,|M_0|)\epsilon}{|M_0|}} + C_4(n,\Lambda_0,|M_0|)\epsilon^\alpha\right).$$

Suppose there is some time $t_0 \in [0,t_1]$ such that

$$\max_{M_{t_0}} |H| > \tilde{\epsilon}. \tag{3.20}$$

Since we also have estimate (3.16) on $|\nabla H|$, and estimate (3.17) on $|\mathring{A}|$, then this assumption (3.20) forces H and $|\mathring{A}|$ to satisfy the inequality in Lemma 3.4 at $t = t_0$. While any closed hypersurface cannot have negative principal curvatures everywhere, each principal curvature λ_i of M_{t_0} will be strictly positive at t_0. In this case, the classic results for the surface area preserving mean curvature flow starting from a strictly convex hypersurface apply and the flow exists for all time after t_0 and converges exponentially to a round sphere ([**McC03**]).

The other possibility is that we have

$$\max_{t\in[0,t_1]} \max_{M_t} |H| \le \tilde{\epsilon}, \tag{3.21}$$

which we will assume from now on.

Choose $\epsilon \le \epsilon_1 = \epsilon_1(n,\Lambda_0,|M_0|)$ sufficiently small so that $\tilde{\epsilon}$ is also sufficiently small. Then by the Michael-Simon's inequality (Lemma 2.12), we have, for any $t \in [0,t_1]$

$$\int_{M_t} H^2 d\mu \le C_5(n,|M_0|)\int_{M_t} |\nabla H|^2 d\mu\,, \tag{3.22}$$

$$\int_{M_t} |\mathring{A}|^2 d\mu \le C_6(n,|M_0|)\int_{M_t} |\nabla \mathring{A}|^2 d\mu\,, \tag{3.23}$$

and

$$\int_{M_t} |\nabla^m A|^2 d\mu \le C_7(n,|M_0|)\int_{M_t} |\nabla^{m+1} A|^2 d\mu\,, \tag{3.24}$$

where Kato's inequalities $|\nabla|\mathring{A}|| \le |\nabla\mathring{A}|$ and $|\nabla|A|| \le |\nabla A|$ are used and Hölder's inequality is used when $n > 2$.

3.3. **Completion of proof of Theorem 3.2.** After above reduction, with (3.21) assumed, we now complete the proof of Theorem 3.2. We first show that $h(t)$ cannot decrease too much in $[0,t_1]$. More specifically,

Lemma 3.5. *There exists $\epsilon_2 = \epsilon_2(n,\Lambda_0,\Lambda_1,|M_0|)$ such that if the conditions in Theorem 3.2 and (3.21) are satisfied for some $\epsilon \le \epsilon_2$, then $h(t) \ge \frac{2}{3\Lambda_1}$ for all $t \in [0,t_1]$.*

Proof. Using (3.22) and the evolution equation of $h(t)$, i.e., Lemma 2.8, we have

$$\begin{aligned}
\frac{dh}{dt} =&\frac{\int_{M_t}[-(1-2hH)(1-hH)|A|^2 + H^2(1-hH)^2 + 2h^2|\nabla H|^2]d\mu}{\int_{M_t} H^2\, d\mu}\\
=&\frac{\int_{M_t}[-(1-3hH+2h^2H^2)(|\mathring{A}|^2+\frac{1}{n}H^2) + H^2(1-hH)^2 + 2h^2|\nabla H|^2]d\mu}{\int_{M_t} H^2\, d\mu}\\
=&\frac{\int_{M_t}[-(1-3hH+2h^2H^2)|\mathring{A}|^2 + \left[\frac{n-1}{n} - (2-\frac{3}{n})hH + \frac{n-2}{n}h^2H^2)\right] H^2]d\mu}{\int_{M_t} H^2\, d\mu}\\
&+\frac{\int_{M_t} 2h^2|\nabla H|^2\, d\mu}{\int_{M_t} H^2\, d\mu}\\
\geq&\frac{\int_{M_t}\left[(3hH-1)|\mathring{A}|^2 - \left[2h^2|\mathring{A}|^2 + (2-\frac{3}{n})hH)\right] H^2 + 2h^2|\nabla H|^2\right] d\mu}{\int_{M_t} H^2\, d\mu}\,.
\end{aligned}$$

Since we have

$$3hH|\mathring{A}|^2 = \frac{3}{2}(2hH|\mathring{A}||\mathring{A}|) \geq -\frac{3}{2}\left(h^2H^2|\mathring{A}|^2 + |\mathring{A}|^2\right),$$

therefore we find

$$\frac{dh}{dt} \geq \frac{\int_{M_t}\left[-\frac{5}{2}|\mathring{A}|^2 - \left[\frac{7}{2}h^2|\mathring{A}|^2 + (2-\frac{3}{n})hH\right] H^2 + 2h^2|\nabla H|^2\right] d\mu}{\int_{M_t} H^2\, d\mu}.$$

Now using (3.6) and (3.21), for any $t \in [0, t_1]$, we have

$$\tilde{\epsilon}^{n-2}\int_{M_t}|H|^2 d\mu \geq \int_{M_t}|H|^2|H|^{n-2}d\mu = \int_{M_t}|H|^n d\mu \geq \omega_n,$$

and so that

$$\int_{M_t}|H|^2 d\mu \geq \omega_n\tilde{\epsilon}^{2-n}\,.$$

Therefore, by (3.12) and (3.22) we have

$$\frac{dh}{dt} \geq -\frac{5}{2\omega_n}\tilde{\epsilon}^{n-2}\int_{M_t}|\mathring{A}|^2 + \frac{\int_{M_t}\left[2h^2 - C_5(n,|M_0|)\left(\frac{7}{2}h^2|\mathring{A}|^2 + \frac{2n+3}{n}|hH|\right)\right]|\nabla H|^2}{\int_{M_t} H^2}.$$

Now using (3.17), (3.21), we can choose $\epsilon_2 = \epsilon_2(n, \Lambda_0, \Lambda_1, |M_0|) > 0$ sufficiently small such that if $\epsilon \leq \epsilon_2$, then

$$2h^2 - C_5(n,|M_0|)\left[\frac{7}{2}h^2|\mathring{A}|^2 + \frac{2n+3}{n}|h||H|\right] \geq 0\,,$$

and consequently we have

$$\frac{dh}{dt} \geq -\int_{M_t}|\mathring{A}|^2\, d\mu \geq -\epsilon^{\frac{\alpha}{2}}. \tag{3.25}$$

Since we have $h(0) \geq \frac{1}{\Lambda_1}$, and the fact that $h(t) \geq \frac{1}{2\Lambda_1}$ for all $t \in [0, t_1]$, we find

$$h(t) \geq \frac{2}{3\Lambda_1} \quad \text{for all } t \in [0, t_1]\,. \tag{3.26}$$

□

Lemma 3.5 negates the possibility of $h(t)$ reaching the extreme value $\frac{1}{2\Lambda_1}$. We now eliminate the other possibility. In other words, as a consequence of Lemma 3.5 and previous estimates, we show that the integral $\int_{M_t} |\nabla^m A|^2 d\mu$ is non-increasing along the flow in $[0, t_1]$:

Corollary 3.6. *There exists $\epsilon_3 = \epsilon_3(n, \Lambda_0, \Lambda_1, |M_0|)$ such that if the conditions in Theorem 3.2 and (3.21) are satisfied for some $\epsilon \leq \epsilon_3$, then we have*

$$\frac{d}{dt} \int_{M_t} |\nabla^m A|^2 d\mu \leq 0$$

for all $t \in [0, t_1]$, and $m \geq 1$.

Proof. (of Corollary 3.6) We first observe that by (3.17) and (3.21), we have:

$$\max_{M_t} |A| \leq \max_{M_t} (|\mathring{A}| + |H|) \leq \sqrt{\frac{C_2(n, \Lambda_0, |M_0|)\epsilon}{|M_0|}} + C_4(n, \Lambda_0, |M_0|)\epsilon^\alpha + \tilde{\epsilon}\,. \tag{3.27}$$

We apply this to the inequality in Lemma 2.11, and use the upper bound for $h(t)$ (3.8) to have:

$$\frac{d}{dt} \int_{M_t} |\nabla^m A|^2 d\mu + 2h \int_{M_t} |\nabla^{m+1} A|^2 d\mu \leq C_8 \epsilon^{\frac{\alpha}{2}} \int_{M_t} |\nabla^m A|^2 d\mu,$$

where $C_8 = C_8(n, m, \Lambda_0, |M_0|)$. We then apply (3.24) to have:

$$\frac{d}{dt} \int_{M_t} |\nabla^m A|^2 d\mu + 2h \int_{M_t} |\nabla^{m+1} A|^2 d\mu \leq C_7 C_8 \epsilon^{\frac{\alpha}{2}} \int_{M_t} |\nabla^{m+1} A|^2 d\mu.$$

Now the conclusion follows from the positive lower bound of $h(t)$, i.e., Lemma 3.5. □

(*continue to the proof of Lemma 3.3*) Corollary 3.6 eliminates the possibility that $\int_{M_{t_1}} |\nabla^m A|^2\, d\mu$ achieves the extreme value $2\Lambda_0$. The other possibility that $h(t_1)$ achieves the extreme value $\frac{1}{2\Lambda_1}$ has been eliminated by Lemma 3.5. Therefore $t_1 \geq t_2 = t_2(n, \Lambda_0, |M_0|)$, and we set $T_1 = t_2(n, \Lambda_0, |M_0|)$ to complete the proof of Lemma 3.3. □

(*completion of the proof of Theorem 3.2*) The only estimate left to establish in Theorem 3.2 is (3.4), but we see that the bound on $|\nabla H|$ is given by (3.16), while the bound on $|\mathring{A}|$ follows from (3.17) by choosing $\alpha < \frac{1}{2}$ if necessary.

□

4. Proof of Theorem 1.1: continued

In the previous section we have shown that if the initial hypersurface is close to a sphere in the L^2-sense (see (3.2)), then either the hypersurface becomes strictly convex at some time, or the estimates (3.3) and (3.4) hold on some time interval $[0, T_1]$. In that proof, we made a key reduction, namely, we showed that it suffices to prove our main theorem when $H(t)$ is close to zero, i.e., (3.21). We will assume that (3.21) holds for the remaining of the argument. More importantly, under this condition on H and (3.17), since $|A|^2 = |\mathring{A}|^2 + \frac{1}{n}H^2$, we find that $|A|$ is uniformly bounded on time interval $[0,T_1]$. Therefore the surface area preserving mean curvature flow (1.1) can be extended pass time T_1 (cf. [**Hui84**]). To prove our main theorem, we only have to address the issues of long-time existence and convergence.

4.1. **Establishing the decay for geometric quantities.** In this subsection, we show that the exponential decay of $\int_{M_t}|\mathring{A}|^2$, assuming (3.21) holds. More precisely, we show:

Theorem 4.1. *Let $M_t^n\subset\mathbb{R}^{n+1}, n\geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow (1.1) on $[0,T_1]$, where T_1 as in Theorem 3.2. Then there exists $\epsilon_4=\epsilon_4(n,\Lambda_0,\Lambda_1,|M_0|)$ such that if the conditions in Theorem 3.2 and (3.21) (with t_1 replaced by T_1) are satisfied for some $\epsilon\leq\epsilon_4$, then for all $t\in[0,T_1]$ we have*

$$\int_{M_t}|\mathring{A}|^2\,d\mu\leq e^{-\delta t}\int_{M_0}|\mathring{A}|^2\,d\mu\,, \tag{4.1}$$

for $\delta=2\Lambda_1C_6(n,|M_0|)>0$, where $C_6(n,|M_0|)$ is from (3.23).

Proof. We use again the evolution equation for $|\mathring{A}|^2$ as in (iii) of Corollary 2.6 and since the surface area is preserved, we have (see also (3.10))

$$\begin{aligned}\frac{d}{dt}\int_{M_t}|\mathring{A}|^2\,d\mu &= \int_{M_t}\frac{\partial}{\partial t}|\mathring{A}|^2d\mu\\ &= -2\int_{M_t}\left[h|\nabla\mathring{A}|^2-h|A|^2|\mathring{A}|^2+\left(\mathrm{tr}(\mathring{A}^3)+\frac{2}{n}H|\mathring{A}|^2\right)\right]d\mu\\ &\leq -2\int_{M_t}\left[h|\nabla\mathring{A}|^2-|\mathring{A}|^2\left(h|A|^2+|\mathring{A}|+\frac{2}{n}|H|\right)\right]d\mu,\end{aligned}$$

where we used that $|\mathrm{tr}(\mathring{A}^3)|\leq|\mathring{A}|^3$. Also note that the flow preserves the surface area so the derivative of the volume vanishes.

Now by (3.3), (3.17) and (3.21) (see also (3.27)), we can choose $\epsilon\leq\epsilon_4$ sufficiently small, where $\epsilon_4=\epsilon_4(n,\Lambda_0,\Lambda_1,|M_0|)$, so that by (3.23), we have

$$\begin{aligned}\frac{d}{dt}\int_{M_t}|\mathring{A}|^2\,d\mu &\leq -2h\int_{M_t}\left[|\nabla\mathring{A}|^2-\frac{1}{2C_6(n,|M_0|)}|\mathring{A}|^2\right]d\mu\\ &\leq -\frac{1}{2\Lambda_1C_6(n,|M_0|)}\int_{M_t}|\mathring{A}|^2\,d\mu.\end{aligned}$$

This completes the proof by setting $\delta=\frac{1}{2\Lambda_1C_6(n,|M_0|)}>0$. ☐

4.2. **Convergence.** In the previous subsection, we obtain the exponential decay for $\int_{M_t}|\mathring{A}|^2\,d\mu$ on some time interval $[0,T_1]$. We now complete the proof for long-time existence of the flow by the following extension theorem:

Theorem 4.2. *Let $M_t^n\subset\mathbb{R}^{n+1}, n\geq 2$, be a smooth compact solution to the surface area preserving mean curvature flow (1.1) with initial condition (3.1). Then there exists $\tilde{\epsilon}_0=\tilde{\epsilon}_0(n,\Lambda_0,\Lambda_1,|M_0|)>0$ and $T_2=T_2(n,\Lambda_0,|M_0|)>0$, such that if*

$$\int_{M_0}|\mathring{A}|^2\,d\mu\leq\epsilon\leq\tilde{\epsilon}_0\,, \tag{4.2}$$

then either at some time $t_0 \in [0, T_1 + T_2]$ the hypersurface M_{t_0} becomes strictly convex and the flow converges exponentially to a round sphere as $t \to \infty$, or for all $t \in [0, T_1 + T_2]$ we have

$$\max\left\{\max_{M_t}|A|, \int_{M_t}|\nabla^m A|^2\,d\mu\right\} \le \Lambda_0 \quad \textit{and} \quad h(t) \ge \frac{1}{2\Lambda_1}\,. \tag{4.3}$$

Proof. By the proof of Theorem 3.2, we know that if $\tilde\epsilon_0 \le \epsilon_0$, then there exists some $T_1 = T_1(n, \Lambda_0, |M_0|) > 0$, such that either at some time $t_0 \in [0, T_1]$ the hypersurface becomes strictly convex and the flow converges exponentially to a round sphere as $t \to \infty$,or for all $t \in [0, T_1]$ we have (see (3.27), Lemma 3.5 and Corollary 3.6) the following:

$$\max_{M_t}|A| \le C\epsilon^\alpha \le \Lambda_0\,, \quad \int_{M_t}|\nabla^m A|^2 d\mu \le \Lambda_0 \quad \text{and} \quad h(t) \ge \frac{1}{2\Lambda_1}. \tag{4.4}$$

Now by Theorem 4.1, if we choose $\tilde\epsilon_0 \le \epsilon_4$ (where ϵ_4 is from Theorem 4.1), then at $t = T_1$ we have

$$\int_{M_{T_1}}|\mathring{A}|^2\,d\mu \le \int_{M_0}|\mathring{A}|^2\,d\mu \le \epsilon\,.$$

Therefore, we can apply Theorem 3.2 to the flow starting at $t = T_1$ with Λ_1 replaced by $2\Lambda_1$ to get $T_2 = T_1(n, \Lambda_0, |M_0|) > 0$. This yields that if the hypersurface does not become strictly convex in $[0, T_1]$, then either at some time $\tilde t_0 \in [T_1, T_1 + T_2]$ the hypersurface becomes strictly convex and the flow converges exponentially to a round sphere as $t \to \infty$, or for all $t \in [T_1, T_1 + T_2]$ we have

$$\max_{M_t}|A| \le \Lambda_0\,, \quad \int_{M_t}|\nabla^m A|^2\,d\mu \le \Lambda_0 \quad \text{and} \quad h(t) \ge \frac{1}{4\Lambda_1}\,. \tag{4.5}$$

On the other hand, in this last case, by (3.25) and Theorem 4.1, for all $t \in [0, T_1 + T_2]$ we have

$$\frac{dh}{dt} \ge -\int_{M_t}|\mathring{A}|^2\,d\mu \ge -e^{-\delta t}\int_{M_0}|\mathring{A}|^2\,d\mu$$

and therefore

$$h(t) \ge h(0) - \epsilon\delta^{-1} \ge \frac{1}{\Lambda_1} - 2\epsilon\Lambda_1 C_6(n, |M_0|)$$

where $C_6(n, |M_0|)$ is from (3.23). Choose $\tilde\epsilon_0$ (possibly smaller) so that

$$h(t) \ge \frac{1}{2\Lambda_1} \quad \text{for all } t \in [0, T_1 + T_2]\,.$$

This together with (4.4), (4.5) yield (4.3). □

We now complete the proof of our main theorem:

Proof. (of Theorem 1.1) Suppose that the initial condition (3.1) is satisfied for some $\Lambda_0, \Lambda_1 > 0$. Then by Theorem 3.2, we choose $\epsilon_0 = \epsilon_0(n, \Lambda_0, \Lambda_1, |M_0|) > 0$ and $T_1 = T_1(n, \Lambda_0, |M_0|) \in (0, 1]$, such that if $\epsilon \le \epsilon_0$, then either the evolving hypersurface becomes strictly convex at some time, or the estimates (3.3) and (3.4) hold for all $t \in [0, T_1]$.

Then we can apply the Theorem 4.2, and we see that if we choose $\epsilon \le \min\{\epsilon_0, \tilde\epsilon_0\}$ then either the flow (1.1) becomes strictly convex at some time $t_0 \in [0, \infty)$ and converges exponentially to a round sphere as $t \to \infty$, or the flow (1.1) exists for all time and the estimates (3.17) and (3.21) holds for all time (so that $|A|$ and $|\nabla^m A|$ are uniformly bounded). Note that in the later case, using Theorem 4.1 we know that the quantity $\int_{M_t} |\mathring{A}|^2 \, d\mu$ decays exponentially, and so that $|\mathring{A}|, |\nabla H|$ and $|\nabla^m A|$ (similar to (3.13) – (3.15)) also decay exponentially. Therefore in the later case the flow also exponentially converges to a round sphere (i.e., $|\mathring{A}| \to 0$ as $t \to \infty$ and the only closed umbilical hypersurface in $\mathbb{R}^{n+1}$ is the round sphere). ☐

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