

CURVATURES ON THE TEICHMÜLLER CURVE

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ABSTRACT. The Teichmüller curve is the fiber space over Teichmüller space T_g of closed Riemann surfaces, where the fiber over a point $(\Sigma, \sigma) \in T_g$ is the underlying surface Σ . We derive formulas for sectional curvatures on the Teichmüller curve. In particular, our method can be applied to investigate the geometry of the Weil-Petersson geodesic as a three-manifold, and the degeneration of the curvatures near the infinity of the augmented Teichmüller space along a Weil-Petersson geodesic, as well as the minimality of hyperbolic surfaces in this 3-manifold.

1. INTRODUCTION

Teichmüller space T_g is the space of hyperbolic metrics on Σ , modulo an equivalent relationship, where two conformal structures σ and ρ are considered equivalent if there is a biholomorphic map between (Σ, σ) and (Σ, ρ) , in the homotopy class of the identity. Here and throughout this paper, we always assume Σ is a smooth, oriented, closed Riemann surface of genus $g > 1$.

Teichmüller space is a complex manifold of complex dimension $3g - 3$ ([Ahl61]), and the cotangent space at a base point (Σ, σ) is identified with $Q(\sigma)$, the space of holomorphic quadratic differentials on (Σ, σ) . Let $\{\phi_1, \dots, \phi_{3g-3}\}$ be a basis of $Q(\sigma)$. The local coordinates of the Teichmüller space T_g in the neighborhood of (Σ, σ) are given by $(t^1, t^2, \dots, t^{3g-3}) \in \mathbb{C}^{3g-3}$. A generic holomorphic quadratic differential ϕdz^2 on (Σ, σ) is written as $\phi dz^2 = \sum_{k=1}^{3g-3} t^k \phi_k dz^2$, where z is the conformal coordinate on (Σ, σ) .

Throughout the paper, σ and ρ are conformal structures, with conformal coordinates z and w , respectively. We also denote $g_\sigma dz d\bar{z}$ and $g_\rho dw d\bar{w}$ as the hyperbolic metrics on (Σ, σ) and (Σ, ρ) , respectively. Corresponding to each ϕdz^2 , there is a hyperbolic metric $g_\rho dw d\bar{w}$ on Σ , from the work of Sampson-Wolf [Sam78, Wlf89]: given a pair of points (two conformal structures) (σ, ρ) in Teichmüller space, there is a unique harmonic map $w : (\Sigma, \sigma) \rightarrow (\Sigma, \rho)$ in the homotopy class of identity, and ϕdz^2 is the Hopf differential of the map u . This induces a homeomorphism $\phi : T_g \rightarrow Q(\sigma)$ which sends ρ to ϕdz^2 . Above correspondence is well-defined via the inverse of this homeomorphism.

The Teichmüller curve \mathcal{T}_g is the fiber space over the Teichmüller space T_g of closed Riemann surfaces, where the fiber over a point $(\Sigma, \sigma) \in T_g$ is the underlying surface Σ . This is a manifold of real dimension $6g - 4$. In this paper, we obtain curvature formulas for a general Riemannian metric on the Teichmüller curve \mathcal{T}_g , and use this to study the geometry of a Riemannian 3-manifold formed by

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a Weil-Petersson geodesic, particularly, the degeneration of its curvatures as the Weil-Petersson geodesic heads towards the boundary of the augmented Teichmüller space.

A point in \mathcal{T}_g is represented as (σ, z_0) , where $\sigma \in T_g$ and z_0 is a point on the marked surface (Σ, σ) . Let $\pi : T\mathcal{T}_g \rightarrow T_g$ be the fibration and the kernel of the differential map $d\pi : T\mathcal{T}_g \rightarrow TT_g$ defines a line bundle ν over the Teichmüller curve \mathcal{T}_g . Wolpert [Wlp86] calculated the Chern form $c_1(\nu)$ of such a line bundle and showed that it is a negative differential 2-form. He suggested to define a Kähler metric G on the Teichmüller curve \mathcal{T}_g such that its Kähler form is $-c_1(\nu)$. The Kähler potential of G is given by $\log \|\frac{\partial}{\partial w}\|$, where $\|\cdot\|$ is the length of \cdot with respect to the hyperbolic metric on a fiber $(\Sigma, \rho(w))$. Here $\rho(w)$ is the conformal structure ρ with the conformal coordinate w . When restricted to each fiber $(\Sigma, \rho(w))$, the metric G is $\frac{1}{2}g_{\rho(w)}dw d\bar{w}$. When evaluated at $\sigma \in T_g$, the metric $G = \frac{1}{2}\sigma dz d\bar{z} + \sum_{k,\ell=1}^{3g-3} \delta_{k\ell} dt^k dt^\ell$. Under this metric G on the Teichmüller curve \mathcal{T}_g , Jost calculated the holomorphic sectional curvature in the fiber direction:

Theorem. [Jos91a] *Under metric G , at the base point (σ, z_0) , the sectional curvature of the tangent plane expanded by $\frac{\partial}{\partial z}$ and $\sqrt{-1}\frac{\partial}{\partial z}$ is*

$$(1.1) \quad K\left(\frac{\partial}{\partial z}, \sqrt{-1}\frac{\partial}{\partial z}\right) = -1 + \sum_{\ell=1}^{3g-3} |\mu_\ell|^2(z_0).$$

where $\{\mu_\ell(z)\}$ is a basis for the tangent space, denoted by $B_h(\sigma)$, of Teichmüller space at (Σ, σ) , normalized according to $D(\mu_\alpha \bar{\mu}_\beta)(z_0) = \delta_{\alpha\beta}$, for the operator $D = -2(\Delta_{g_\sigma} - 2)^{-1}$ on the hyperbolic surface $(\Sigma, g_\sigma dz d\bar{z})$.

Note that the tangent space $B_h(\sigma)$ of Teichmüller space at (Σ, σ) is the space of so-called *harmonic Beltrami differentials*, in the form of $\mu(z) \frac{d\bar{z}}{dz} = \frac{\phi d\bar{z}^2}{g_\sigma dz d\bar{z}}$ for some $\phi dz^2 \in Q(\sigma)$. In the present work, using a general Riemannian metric on \mathcal{T}_g , we find the same curvature formula holds in the fiber directions. Moreover, we determine the sectional curvatures of the directions spanned by one fiber direction and the other by a tangent vector in Teichmüller space.

Denote $z = x + \sqrt{-1}y$ and $t^\ell = x^\ell + \sqrt{-1}y^\ell$. We consider a Riemannian metric G' on the Teichmüller curve \mathcal{T}_g :

$$(1.2) \quad G' = g_{\rho(w)} dw d\bar{w} + \sum h_{\alpha\beta}(w) d\nu^\alpha d\nu^\beta$$

where $\nu^\alpha, \nu^\beta \in \{x^1, y^1, \dots, x^{3g-3}, y^{3g-3}\}$. The metric G' restricted on the fiber (Σ, ρ) is identified with the hyperbolic metric $g_{\rho(w)} dw d\bar{w}$.

Theorem 1.1. *With respect to the metric G' on the Teichmüller curve \mathcal{T}_g , we assume functions $\{h_{\alpha\beta}\}$ satisfy the following:*

$$(1.3) \quad \sum h_{\alpha\beta}(z) d\nu^\alpha d\nu^\beta = 2 \sum_{\ell=1}^{3g-3} ((dx^\ell)^2 + (dy^\ell)^2), \forall z \in (\Sigma, \sigma),$$

i.e., it is Euclidean when restricted on (Σ, σ) in \mathcal{T}_g . Then at the base point (σ, z_0) , the sectional curvature of the tangent plane expanded by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ is

$$(1.4) \quad K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -1 + \sum_{\ell=1}^{3g-3} |\mu_\ell|^2(z_0),$$

where $\mu_\ell = \frac{\bar{\phi}_\ell}{g_\sigma}$ and the basis $\{\phi_\ell dz^2\}$ on $Q(\sigma)$ is chosen such that (1.3) holds.

If we require further that functions $\{h_{\alpha\beta}\}$ satisfy the following:

$$(1.5) \quad \sum h_{\alpha\beta}(w) d\nu^\alpha d\nu^\beta = 2 \sum_{\ell=1}^{3g-3} ((dx^\ell)^2 + (dy^\ell)^2), \forall \rho(w) \in T_g,$$

i.e., a local Euclidean structure rather than a single point one as in (1.3), then we have:

Theorem 1.2. *With respect to the metric G' on the Teichmüller curve \mathcal{T}_g , we choose the basis $\{\phi_\ell dz^2\}$ on $Q(\sigma)$ such that (1.5) holds for functions $\{h_{\alpha\beta}\}$ in G' . Let $z = x + \sqrt{-1}y$ be the coordinate of a fiber and $\{t^\ell = x^\ell + \sqrt{-1}y^\ell\}_{\ell=1}^{3g-3}$ be the coordinates of the Teichmüller space. Then at the base point (σ, z_0) , the sectional curvatures of the tangent planes expanded by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x^\ell}$, $\ell = 1, 2, \dots, 3g-3$, are:*

$$(1.6) \quad K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x^\ell}\right) = -\frac{1}{2}D(|\mu_\ell|^2)(z_0),$$

where $\mu_\ell = \frac{\bar{\phi}_\ell}{g_\sigma}$. Similarly,

$$(1.7) \quad K\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x^\ell}\right) = K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y^\ell}\right) = K\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y^\ell}\right) = -\frac{1}{2}D(|\mu_\ell|^2)(z_0).$$

They are all non-positive.

One of the motivations of this paper is to study the Weil-Petersson geodesics in Teichmüller space from its intrinsic geometry. As conformal structures of a topological surface travel along a curve in Teichmüller space, they form a three-space which is homeomorphic to $\Sigma \times \mathbb{R}$. We would like to set up as follows to study the shape of a Weil-Petersson geodesic: we take one tangent vector $\mu_0 \frac{d\bar{z}}{dz}$ at the point $\sigma \in T_g$. Consider the Weil-Petersson geodesic $\gamma = \gamma(s)$ in Teichmüller space T_g through the point σ and in the direction $\mu_0 \frac{d\bar{z}}{dz}$. We consider the germ N_σ at σ , and a natural local metric denoted by H , near $t = 0$, takes a very simple form:

$$(1.8) \quad H = g_{\rho(w)} dwd\bar{w} + dt^2,$$

where $g_{\rho(w)} dwd\bar{w}$ is the pull-back metric will be made clear in (3.1).

Theorem 1.3. *At any point $z_0 \in (\Sigma, \sigma)$, we determine the sectional curvatures of the tangent planes spanned by two of the three vectors $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial t}$, with respect to the metric H on a Weil-Petersson geodesic γ through σ in the direction of $\mu_0 \frac{d\bar{z}}{dz} \in B_h(\sigma)$, where there is a $\phi_0 dz^2 \in Q(\sigma)$ such that $\mu_0 \frac{d\bar{z}}{dz} = \frac{\bar{\phi}_0 dz^2}{g_\sigma |dz|^2}$:*

- (1) $K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)(\sigma, z_0) = -1 + |\mu_0(z_0)|^2$;
- (2) $K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)(\sigma, z_0) = K\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right)(\sigma, z_0) = -D(|\mu_0|^2)(z_0)$.

Moreover, we define a Riemannian structure on the surface bundle $N = \bigcup_{\sigma \in \gamma} N_\sigma$ over the Weil-Petersson geodesic γ , and find

Theorem 1.4. *Let γ be a Weil-Petersson geodesic in Teichmüller space. Then there exists a choice of a metric in each conformal class along γ , such that the surface bundle N over γ acquires a Riemannian metric whose germ at each fiber is N_σ as in (1.8). In particular,*

- (1) each fiber over σ is a hyperbolic surface;
- (2) each fiber over σ is also a minimal surface;

(3) *the sectional curvatures of the metric are given in the Theorem 1.3.*

Generally, it is difficult to carry out calculations on the Weil-Petersson metric on Teichmüller space. Part of the reason is that the metric itself, as well as the curvature tensor formula, are given as an integral (or a sum of integrals) over the underlying conformal structure (Σ, σ) , rather than in terms of any global coordinates of Teichmüller space. In this work, the technical tool we intensively rely on is the Teichmüller theory of harmonic maps, begun with the pioneer work of Eells-Sampson ([ES64]). This theory has many applications in many different areas in geometry and analysis. In the case of compact hyperbolic surfaces, the analytical theory does not involve any regularity issue: the degree one harmonic map between hyperbolic surfaces is a diffeomorphism ([SY78]). It has become an important computational tool in Teichmüller theory, as well as hyperbolic geometry (see, for example, [Wlf89], [Jos91a, Jos91b, JY09], [Min92b, Min92a], [Hua05, Hua07]).

Plan of the paper. We start with a brief collection of preliminary facts in §2, where we introduce harmonic maps between hyperbolic surface in §2.1, Teichmüller space and the Teichmüller curve in §2.2, and the Weil-Petersson metric in §2.3. Theorems 1.1 and 1.2 are proved in §3, where we obtain formulas for the curvatures of the Teichmüller curve for the Riemannian metric G' ((1.2)). We will focus on the geometry of the Weil-Petersson geodesic in the last section §4, prove the Theorems 1.3 and 1.4.

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2. BACKGROUND

2.1. Harmonic maps between surfaces. We review our basic computational scheme, which is based on harmonic maps between compact hyperbolic surfaces.

Let $w : (\Sigma, g_\sigma |dz|^2) \rightarrow (\Sigma, g_\rho |dw|^2)$ be a Lipschitz map, where $g_\sigma |dz|^2$ and $g_\rho |dw|^2$ are hyperbolic metrics on the surface Σ , associated to the conformal structures σ and ρ , respectively. And z and w are conformal coordinates on (Σ, σ) and (Σ, ρ) . We (following [Sam78]) define important density functions: the *holomorphic energy density*

$$(2.1) \quad \mathcal{H}(z) = \frac{g_\rho}{g_\sigma} |w_z|^2,$$

and the *anti-holomorphic energy density*

$$(2.2) \quad \mathcal{L}(z) = \frac{g_\rho}{g_\sigma} |w_{\bar{z}}|^2.$$

The energy density function of the map w is now simply

$$(2.3) \quad e(w(z)) = \mathcal{H}(z) + \mathcal{L}(z),$$

and the *total energy* and the *Jacobian determinant* of the map are given by

$$(2.4) \quad E(w, \sigma, \rho) = \int_{\Sigma} e(z) g_\sigma |dz|^2 = \int_{\Sigma} e(z) dA$$

and

$$(2.5) \quad \mathcal{J}(z) = \mathcal{H}(z) - \mathcal{L}(z),$$

respectively. Here dA in (2.3) is the area element for (Σ, σ) .

The map w is *harmonic* if it is a critical point of this total energy functional (2.4). The $(2, 0)$ part of the pullback $w^*\rho$ is particularly important, and it is called *Hopf differential* of w :

$$\phi(z)dz^2 = (w^*\rho)^{(2,0)} = g_\rho w_z \bar{w}_z dz^2.$$

It is well-known that there is a unique harmonic map $w : (\Sigma, \sigma) \rightarrow (\Sigma, \rho)$ in each homotopy class, and w is a diffeomorphism with positive Jacobian determinant.

2.2. Teichmüller space and the Teichmüller curve. For a closed surface Σ of genus $g > 1$, it is a consequence of the Uniformization Theorem that the notion of conformal structures, complex structures, and hyperbolic metrics on Σ are equivalent. Teichmüller space T_g is the space of conformal structures on Σ , modulo the group of orientation preserving diffeomorphisms isotopic to the identity.

For a conformal structure σ on Σ , it represents a point in T_g , and we denote by z its conformal coordinate. We routinely use (Σ, σ) to indicate the marking by a conformal structure. The cotangent space of T_g at σ is identified as $Q(\sigma) = \{\phi(z)dz^2 : \bar{\partial}\phi = 0\}$, the space of holomorphic quadratic differentials on (Σ, σ) . The Hopf differential of a harmonic map is holomorphic, hence belongs to $Q(\sigma)$, and the map w is conformal if and only if its Hopf differential is 0. This is essentially the entrance of harmonic map theory to Teichmüller theory. Note that $Q(\sigma)$ is a Banach space of complex dimension $3g - 3$, while the space of Beltrami differentials is of infinite dimension.

We denote $g_\sigma|dz|^2$ the hyperbolic metric on (Σ, σ) , and its Laplacian is

$$\Delta_\sigma = \frac{4}{g_\sigma} \frac{\partial^2}{\partial z \partial \bar{z}},$$

with nonpositive eigenvalues. We also define an operator $D = -2(\Delta_\sigma - 2)^{-1}$. It is L^2 -self-adjoint with respect to $(\Sigma, g_\sigma|dz|^2)$. This operator plays an essential role in understanding the Weil-Petersson geometry of Teichmüller space.

The tangent space of T_g at σ can be identified with the space of harmonic Beltrami differentials $B_h(\sigma)$. A Beltrami differential $\mu(z) \frac{d\bar{z}}{dz}$ on (Σ, σ) is harmonic if $\mu(z) \frac{d\bar{z}}{dz} = \frac{\bar{\phi} d\bar{z}^2}{g_\sigma dz d\bar{z}}$ for some $\phi dz^2 \in Q(\sigma)$.

The Teichmüller curve \mathcal{T}_g is a fiber bundle over Teichmüller space T_g , and the fiber over $\sigma \in T_g$ is the marked surface (Σ, σ) . As a manifold of real dimension $6g - 4$, every point in \mathcal{T}_g can be represented as (σ, z_0) , where $\sigma \in T_g$ and $z_0 \in (\Sigma, \sigma)$.

The first Chern class $c_1(\nu) = \frac{\sqrt{-1}}{2\pi} \Theta$ of the line bundle ν over \mathcal{T}_g is computed by Wolpert ([Wlp86]), where the curvature 2-form Θ is found to satisfy $\Theta(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = \frac{-2}{(z - \bar{z})^2}$, and $\Theta(\frac{\partial}{\partial \bar{z}}, \tau_\mu) = 0$, and $\Theta(\bar{\tau}_\nu, \tau_\mu) = D(\mu\bar{\nu})$. It is negative, therefore one can define a Kähler metric $-c_1(\nu)$ on \mathcal{T}_g .

2.3. The Weil-Petersson metric and 3-manifolds. The Weil-Petersson co-metric is defined on the cotangent space $Q(\sigma)$ by the natural L^2 -norm:

$$(2.6) \quad \|\phi\|_{WP}^2 = \int_\Sigma \frac{|\phi|^2}{g_\sigma^2} dA_\sigma, \forall \phi \in Q(\sigma),$$

where dA_σ is the hyperbolic area element of (Σ, σ) . By duality, we obtain the Weil-Petersson metric on T_g .

The Weil-Petersson geodesics are intimately related to the geometry of three manifolds. Given $X, Y \in T_g(\Sigma)$, they uniquely determine a quasi-Fuchsian hyperbolic 3-manifold, $Q(X, Y)$, with X and Y as conformal boundaries, by the means of Bers' simultaneous uniformization ([Ber72]). Brock ([Bro03]) showed that the hyperbolic volume of the convex core of $Q(X, Y)$ is quasi-isometric to the length of the Weil-Petersson geodesic joining X and Y in Teichmüller space. We obtained a Weil-Petersson potential from varying quasi-Fuchsian manifolds in quasi-Fuchsian space near the Fuchsian locus ([GHW09]).

Many mysterious properties of the Weil-Petersson geometry of Teichmüller space are largely due to the incompleteness of the metric ([Chu76, Wlp75]). When a Weil-Petersson geodesic can not be extended, a short simple closed curve on the surface is pinched to a single point ([Mas76]), while curvatures on the surface remain hyperbolic. A basic property is that the Weil-Petersson metric is geodesically convex ([Wlp87]), hence any two points can be joined by a unique Weil-Petersson geodesic. The Weil-Petersson geometry of Teichmüller space is quite satisfying: it is a space of negative curvature ([Tro86, Wlp86]). However, there is neither negative upper bound ([Hua05]) nor lower bound ([Hua07, Sch86]) of the sectional curvatures. We refer more detailed discussions on the Weil-Petersson geometry of Teichmüller space to a series of articles ([Wlp03, Wlp06, Wlp09, Wlp10b]) and the book [Wlp10a].

We want to understand the infinitesimal geometry of the Weil-Petersson geodesic. To this end, we consider a Weil-Petersson geodesic γ determined by the point $\sigma \in T_g$ and the direction μ_ℓ , where $\mu_\ell \in B_h(\sigma)$ is a tangent vector at σ . We denote N_σ the germ of a hyperbolic surface associated to the conformal structure at the point $\sigma \in \gamma$. The set of these germs over the Weil-Petersson geodesic γ form a three-dimensional space.

3. CURVATURE FORMULAS OF THE TEICHMÜLLER CURVE

We prove theorems 1.1 and 1.2 in this section. Curvatures in the fiber directions are obtained in §3.1, and fiber-tangential directions follow in §3.2.

3.1. Fiber directions. In this subsection, we determine the sectional curvature of the Teichmüller curve \mathcal{T}_g in the fiber directions. The metric G' is given as in (1.2):

$$G' = g_{\rho(w)}dw d\bar{w} + \sum h_{\alpha\beta}(w)d\nu^\alpha d\nu^\beta,$$

where the functions $\{h_{\alpha\beta}\}$ satisfy (1.3):

$$\sum h_{\alpha\beta}(z)d\nu^\alpha d\nu^\beta = 2 \sum_{\ell=1}^{3g-3} ((dx^\ell)^2 + (dy^\ell)^2), \forall z \in (\Sigma, \sigma).$$

In other words, we only require it to be Euclidean at $w = z$ in T_g .

We now proceed to calculate the sectional curvature in the fiber directions, spanned by vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, where $z = x + \sqrt{-1}y$.

Proof of Theorem 1.1. Fixing $\sigma \in T_g$, for any hyperbolic metric g_ρ on Σ , denote the unique harmonic map $w : (\Sigma, \sigma) \rightarrow (\Sigma, g_\rho dw d\bar{w})$ in the homotopy class of the identity, one obtains the Hopf differential $\phi(z)dz^2$ of w , which is holomorphic. Therefore we have a map ϕ from Teichmüller space to $Q(\sigma)$. This map is a homeomorphism ([Sam78, Wlf89]). Therefore, via the inverse map, $Q(\sigma)$ provides global coordinates for T_g .

The space of holomorphic quadratic differentials $Q(\sigma)$ is a Banach space, so for $\phi_0 dz^2 \in Q(\sigma) \setminus \{0\}$, and $\phi(t) = t\phi_0$ is a ray in $Q(\sigma)$. We denote the hyperbolic metrics $g_{\rho(t)}|dw(t)|^2$ as the points in T_g determined by the ray $\phi(t)$ in $Q(\sigma)$, via the Sampson-Wolf Theorem. Here $w(t)$ is the family of harmonic maps, in the homotopy class of the identity, whose associated Hopf differentials are given by $\phi(t) = \rho(w(t))w_z(t)\bar{w}_z(t)dz^2$.

Clearly, at $t = 0$, $\rho(0) = \sigma$.

For this family of harmonic maps $w(t) : (\Sigma, g_\sigma|dz|^2) \rightarrow (\Sigma, g_{\rho(t)}|dw(t)|^2)$, we pull back the metric to find

$$\begin{aligned} w^*g_{\rho(t)}|dw(t)|^2 &= 2\operatorname{Re}(g_{\rho(w(t))}w_z(t)\bar{w}_z(t)dz^2) + g_{\rho(t)}(|w_z(t)|^2 + |\bar{w}_z(t)|^2)|dz|^2 \\ (3.1) \quad &= \phi(t)dz^2 + g_\sigma e(t)|dz|^2 + \bar{\phi}(t)d\bar{z}^2, \end{aligned}$$

where $e = e(t)$ is the energy density of $w(t)$, as in (2.3).

For $\phi(t)dz^2$, we have $\mu(t)\frac{d\bar{z}}{dz} = \frac{\bar{\phi}(t)dz^2}{g_\sigma dzd\bar{z}} \in B_h(\sigma)$, a family of harmonic Beltrami differentials. They represent tangent vectors at $\sigma \in T_g$.

Let $\{\phi_1, \dots, \phi_{3g-3}\}$ be a basis of $Q(\sigma)$ such that (1.3) holds for the functions $\{h_{\alpha\beta}\}$ in the definition (1.2) of the metric G' of \mathcal{T}_g .

We write $t = (t^1, \dots, t^{3g-3})$ such that $\phi(t)dz^2 = \sum_{\ell=1}^{3g-3} t^\ell \phi_\ell dz^2$. And we use $|_0$ to indicate evaluation at $t^\alpha = 0$ for any $\alpha = 1, \dots, 3g-3$.

The variations of the energy density $e(t)$ are calculated by Wolf ([Wlf89]) as follows:

$$\begin{aligned} e(t)|_0 &= 1, \\ \frac{\partial e(t)}{\partial t^\alpha}|_0 &= \frac{\partial e(t)}{\partial t^\alpha}|_0 = 0, \\ \frac{\partial^2 e(t)}{\partial t^\alpha \partial t^\beta}|_0 &= \frac{\partial^2 e(t)}{\partial t^\alpha \partial t^\beta}|_0 = 0, \\ \frac{\partial^2 e(t)}{\partial t^\alpha \partial t^\beta}|_0 &= (D+1) \frac{\phi_\alpha \bar{\phi}_\beta}{g_\sigma^2}. \end{aligned}$$

Using real coordinates, we can rewrite the above results as:

$$\begin{aligned} e(t)|_0 &= 1, \\ \frac{\partial e(t)}{\partial x^\alpha}|_0 &= \frac{\partial e(t)}{\partial y^\alpha}|_0 = 0, \\ \frac{\partial^2 e(t)}{\partial x^\alpha \partial x^\beta}|_0 &= \frac{\partial^2 e(t)}{\partial y^\alpha \partial y^\beta}|_0 = (D+1) \frac{2\operatorname{Re}(\phi_\alpha \bar{\phi}_\beta)}{g_\sigma^2}, \end{aligned}$$

Using the pullback (3.1), the metric G' on \mathcal{T}_g is written as

$$\begin{aligned} (3.2) \quad G' &= \phi(t)dz^2 + g_\sigma e(t)dzd\bar{z} + \bar{\phi}(t)d\bar{z}^2 + \sum h_{\alpha\beta}(w)dv^\alpha dv^\beta \\ &= (g_\sigma e + 2\operatorname{Re}\phi)dx^2 - 4(\operatorname{Im}\phi)dxdy + (g_\sigma e - 2\operatorname{Re}\phi)dy^2 + \sum h_{\alpha\beta}(w)dv^\alpha dv^\beta. \end{aligned}$$

To simplify our notation, we denote $R_{1221} = R_{xyyx}$, and utilize Einstein notation to compute this curvature tensor as follows:

$$\begin{aligned}
R_{1221}|_0 &= g_{11}(\Gamma_{22,1}^1 - \Gamma_{12,2}^1 + \Gamma_{22}^\beta \Gamma_{1\beta}^1 - \Gamma_{12}^\beta \Gamma_{2\beta}^1)|_0 \\
&= g_{11}(\Gamma_{22,1}^1 - \Gamma_{12,2}^1 \\
&\quad + \Gamma_{22}^1 \Gamma_{11}^1 + \Gamma_{22}^2 \Gamma_{12}^1 + \sum_{\ell=1}^{3g-3} (\Gamma_{22}^{x^\ell} \Gamma_{1x^\ell}^1 + \Gamma_{22}^{y^\ell} \Gamma_{1y^\ell}^1) \\
(3.3) \quad &\quad - \Gamma_{12}^1 \Gamma_{21}^1 - \Gamma_{12}^2 \Gamma_{22}^1 - \sum_{\ell=1}^{3g-3} (\Gamma_{12}^{x^\ell} \Gamma_{2x^\ell}^1 + \Gamma_{12}^{y^\ell} \Gamma_{2y^\ell}^1))|_0.
\end{aligned}$$

Here we recall from (1.3) that, at $t = 0$, the functions $h_{\alpha\beta}$ satisfy the condition that $\sum h_{\alpha\beta}(z) d\nu^\alpha d\nu^\beta = 2 \sum_{\ell=1}^{3g-3} ((dx^\ell)^2 + (dy^\ell)^2)$.

We calculate the values of Christoffel symbols evaluated at $t^\alpha = x^\alpha + \sqrt{-1}y^\alpha = 0$ for any α :

$$\left\{ \begin{array}{l} \Gamma_{11}^1 = \frac{(g_\sigma)_1}{2g_\sigma}, \quad \Gamma_{11}^2 = -\frac{(g_\sigma)_2}{2g_\sigma}, \quad \Gamma_{11}^{x^\ell} = -\frac{Re\phi_\ell}{2}, \quad \Gamma_{11}^{y^\ell} = \frac{Im\phi_\ell}{2}, \\ \Gamma_{12}^1 = \frac{(g_\sigma)_2}{2\sigma}, \quad \Gamma_{12}^2 = \frac{(g_\sigma)_1}{2\sigma}, \quad \Gamma_{12}^{x^\ell} = \frac{Im\phi_\ell}{2}, \quad \Gamma_{12}^{y^\ell} = \frac{Re\phi_\ell}{2}, \\ \Gamma_{22}^1 = -\frac{(g_\sigma)_1}{2g_\sigma}, \quad \Gamma_{22}^2 = \frac{(g_\sigma)_2}{2g_\sigma}, \quad \Gamma_{22}^{x^\ell} = \frac{Re\phi_\ell}{2}, \quad \Gamma_{22}^{y^\ell} = -\frac{Im\phi_\ell}{2}. \end{array} \right.$$

Some mixed terms are calculated as follows:

$$\left\{ \begin{array}{l} \Gamma_{1x^\ell}^1 = \frac{Re\phi_\ell}{g_\sigma}, \quad \Gamma_{1x^\ell}^2 = -\frac{Im\phi_\ell}{g_\sigma}, \quad \Gamma_{1x^\ell}^{x^\ell} = 0, \quad \Gamma_{1x^\ell}^{y^\ell} = 0, \\ \Gamma_{1y^\ell}^1 = -\frac{Im\phi_\ell}{g_\sigma}, \quad \Gamma_{1y^\ell}^2 = -\frac{Re\phi_\ell}{g_\sigma}, \quad \Gamma_{1y^\ell}^{x^\ell} = 0, \quad \Gamma_{1y^\ell}^{y^\ell} = 0, \\ \Gamma_{2x^\ell}^1 = -\frac{Im\phi_\ell}{g_\sigma}, \quad \Gamma_{2x^\ell}^2 = -\frac{Re\phi_\ell}{g_\sigma}, \quad \Gamma_{2x^\ell}^{x^\ell} = 0, \quad \Gamma_{2x^\ell}^{y^\ell} = 0, \\ \Gamma_{2y^\ell}^1 = -\frac{Re\phi_\ell}{g_\sigma}, \quad \Gamma_{2y^\ell}^2 = \frac{Im\phi_\ell}{g_\sigma}, \quad \Gamma_{2y^\ell}^{x^\ell} = 0, \quad \Gamma_{2y^\ell}^{y^\ell} = 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \Gamma_{x^\ell x^\ell}^1 = 0, \quad \Gamma_{x^\ell x^\ell}^2 = 0, \\ \Gamma_{x^\ell y^\ell}^1 = 0, \quad \Gamma_{x^\ell y^\ell}^2 = 0, \\ \Gamma_{y^\ell y^\ell}^1 = 0, \quad \Gamma_{y^\ell y^\ell}^2 = 0. \end{array} \right.$$

We calculate the first derivative of the Christoffel symbols in the curvature tensor R_{1221} by changing the order of the derivative respect to x, y and the evaluation at $x^\alpha = y^\alpha = 0, \alpha = 1, \dots, 3g-3$.

$$\Gamma_{22,1}^1|_0 = (\Gamma_{22}^1|_0)_1 = -\left(\frac{(g_\sigma)_1}{2g_\sigma}\right)_1,$$

$$\Gamma_{12,2}^1|_0 = (\Gamma_{12}^1|_0)_2 = \left(\frac{(g_\sigma)_2}{2g_\sigma}\right)_2.$$

Using that the curvature of the hyperbolic metric $g_\sigma(dx^2+dy^2)$ is -1 , and $g_{11}|_0 = g_\sigma$ from (3.2), we have:

$$\begin{aligned} R_{1221}|_0 &= g_\sigma\left(-\left(\frac{(g_\sigma)_1}{2g_\sigma}\right)_1 - \left(\frac{(g_\sigma)_2}{2g_\sigma}\right)_2\right. \\ &\quad \left.- \frac{((g_\sigma)_1)^2}{4g_\sigma^2} + \frac{((g_\sigma)_2)^2}{4g_\sigma^2} + \frac{1}{2} \sum_{\ell=1}^{3g-3} \left(\frac{(Re\phi_\ell)^2}{g_\sigma} + \frac{(Im\phi_\ell)^2}{g_\sigma}\right)\right. \\ &\quad \left.- \frac{((g_\sigma)_2)^2}{4g_\sigma^2} + \frac{((g_\sigma)_1)^2}{4g_\sigma^2} + \frac{1}{2} \sum_{\ell=1}^{3g-3} \left(\frac{(Re\phi_\ell)^2}{g_\sigma} + \frac{(Im\phi_\ell)^2}{g_\sigma}\right)\right) \\ &= g_\sigma\left(-\left(\frac{(g_\sigma)_1}{2g_\sigma}\right)_1 - \left(\frac{(g_\sigma)_2}{2g_\sigma}\right)_2 + \frac{1}{g_\sigma} \sum_{\ell=1}^{3g-3} |\phi_\ell|^2\right) \\ &= -g_\sigma^2 + \sum_{\ell=1}^{3g-3} |\phi_\ell|^2. \end{aligned}$$

Therefore the curvature in directions $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, at $t = 0$, is

$$\begin{aligned} K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= \frac{R_{1221}}{g_\sigma^2} = -1 + \sum_{\ell=1}^{3g-3} \frac{|\phi_\ell|^2}{g_\sigma^2}(z_0) \\ &= -1 + \sum_{\ell=1}^{3g-3} |\mu_\ell|^2(z_0). \end{aligned}$$

□

The normalization for the metric $G = -c_1(v)$ on \mathcal{T}_g provides the following estimates of the curvatures in the fiber directions, in particular, it is impossible for the curvatures to be non-positive everywhere in \mathcal{T}_g :

Corollary 3.1. *With respect to the metric $G = -c_1(v)$ on \mathcal{T}_g , we have*

- (1) [Jos91a]: $Sup_{(\sigma, z_0) \in \mathcal{T}_g} K\left(\frac{\partial}{\partial z}, \sqrt{-1} \frac{\partial}{\partial z}\right) > 0$;
- (2) $Sup_{(\sigma, z_0) \in \mathcal{T}_g} K\left(\frac{\partial}{\partial z}, \sqrt{-1} \frac{\partial}{\partial z}\right) \leq 9g - 10$.

Proof.

- (1) This is proved in ([Jos91a]). The idea is that the explicit Kähler potential of the metric G forces $D(|\mu_\ell|^2)(z_0) = 1$ for all $\ell = 1, \dots, 3g - 3$, then the lower bound 0 is a consequence of the maximum principle.
- (2) The upper bound is a consequence of the following point-wise estimate from next Lemma.

□

Lemma 3.2. [Wlf08] *For any $\mu(z) \frac{d\bar{z}}{dz} \in B_h(\sigma)$, and $\forall z \in (\Sigma, \sigma)$, we have the following:*

$$3D(|\mu|^2)(z) \geq |\mu|^2(z).$$

3.2. Fiber-tangential directions. To calculate the sectional curvatures spanned by one fiber direction and one tangential direction, we have to complete the mixed terms in Christoffel symbols which were not required in the last subsection. For this reason, we require functions $\{h_{\alpha\beta}\}$ satisfy (1.5):

$$\sum h_{\alpha\beta}(w)dv^\alpha dv^\beta = 2 \sum_{\ell=1}^{3g-3} ((dx^\ell)^2 + (dy^\ell)^2), \forall \rho(w) \in T_g.$$

Therefore the metric G' takes the form:

$$(3.4) \quad G' = g_{\rho(w)}dw d\bar{w} + 2 \sum_{\ell=1}^{3g-3} (dx_\ell^2 + dy_\ell^2).$$

And we are now going to prove the Theorem 1.2.

Proof of Theorem 1.2. We will continue to use notation in the proof of Theorem 1.1. We will only calculate the curvature tensor $R_{1x^\ell x^\ell 1} = R_{xx^\ell x^\ell x}$ at $t = 0$, which is given by:

$$(3.5) \quad R_{1x^\ell x^\ell 1}|_0 = g_\sigma(\Gamma_{x^\ell x^\ell, 1}^1 - \Gamma_{1x^\ell, x^\ell}^1 + \Gamma_{x^\ell x^\ell}^\beta \Gamma_{1\beta}^1 - \Gamma_{1x^\ell}^\beta \Gamma_{x^\ell \beta}^1)|_0.$$

We have

$$\begin{aligned} \Gamma_{1x^\ell}^1 &= \frac{1}{2} \frac{g_\sigma e - 2Re\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (g_\sigma \frac{\partial e}{\partial x^\ell} + 2Re\phi_\ell) + \frac{1}{2} \frac{2Im\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (-2Im\phi_\ell), \\ \Gamma_{1y^\ell}^1 &= \frac{1}{2} \frac{g_\sigma e - 2Re\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (g_\sigma \frac{\partial e}{\partial y^\ell} - 2Im\phi_\ell) + \frac{1}{2} \frac{2Im\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (2Re\phi_\ell), \\ \Gamma_{2x^\ell}^2 &= \frac{1}{2} \frac{2Im\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (-2Im\phi_\ell) + \frac{1}{2} \frac{g_\sigma e + 2Re\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (g_\sigma \frac{\partial e}{\partial x^\ell} - 2Re\phi_\ell), \\ \Gamma_{2y^\ell}^2 &= \frac{1}{2} \frac{2Im\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (2Re\phi_\ell) + \frac{1}{2} \frac{g_\sigma e + 2Re\phi}{g_\sigma^2 e^2 - 4|\phi|^2} (g_\sigma \frac{\partial e}{\partial y^\ell} + 2Im\phi_\ell). \end{aligned}$$

And their derivatives at $t = 0$ are as follows:

$$(3.6) \quad \Gamma_{1x^\ell, x^\ell}^1|_0 = \Gamma_{2x^\ell, x^\ell}^1|_0 = -2 \frac{|\phi_\ell|^2}{g_\sigma^2} + \frac{1}{2} \frac{\partial^2 e(w)}{\partial x^\ell \partial x^\ell} = (D-1) \frac{|\phi_\ell|^2}{g_\sigma^2},$$

$$(3.7) \quad \Gamma_{1y^\ell, y^\ell}^1|_0 = \Gamma_{2y^\ell, y^\ell}^1|_0 = -2 \frac{|\phi_\ell|^2}{g_\sigma^2} + \frac{1}{2} \frac{\partial^2 e(w)}{\partial y^\ell \partial y^\ell} = (D-1) \frac{|\phi_\ell|^2}{g_\sigma^2}.$$

Moreover, because of condition (3.4), we can complete the table of Christoffel symbols by:

$$\left\{ \begin{array}{l} \Gamma_{x^\ell x^\ell}^x = 0, \quad \Gamma_{x^\ell x^\ell}^{y^\ell} = 0, \\ \Gamma_{x^\ell y^\ell}^x = 0, \quad \Gamma_{x^\ell y^\ell}^{y^\ell} = 0, \\ \Gamma_{y^\ell y^\ell}^x = 0, \quad \Gamma_{y^\ell y^\ell}^{y^\ell} = 0. \end{array} \right.$$

Substituting these terms into (3.5), we find

$$\begin{aligned} R_{1x^\ell x^{\ell 1}}|_0 &= g_\sigma(\Gamma_{x^\ell x^{\ell 1}}^1 - \Gamma_{1x^\ell, x^\ell}^1 + \Gamma_{x^\ell x^\ell}^\beta \Gamma_{1\beta}^1 - \Gamma_{1x^\ell}^\beta \Gamma_{x^\ell \beta}^1)|_0 \\ &= g_\sigma(0 - (D-1)\frac{|\phi_\ell|^2}{g_\sigma^2} + 0 - \frac{(Re\phi_\ell)^2}{g_\sigma^2} - \frac{(Im\phi_\ell)^2}{g_\sigma^2}) \\ &= -g_\sigma D\left(\frac{|\phi_\ell|^2}{g_\sigma^2}\right)(z_0). \end{aligned}$$

Then the curvature

$$K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x^\ell}\right) = \frac{R_{1x^\ell x^{\ell 1}}}{2g_\sigma} = -\frac{1}{2}D(|\mu_\ell|^2)(z_0).$$

Similarly, one can work out other curvatures:

$$\frac{R_{1y^\ell y^{\ell 1}}}{2g_\sigma} = \frac{R_{2x^\ell x^{\ell 2}}}{2g_\sigma} = \frac{R_{2y^\ell y^{\ell 2}}}{2g_\sigma} = -\frac{1}{2}D(|\mu_\ell|^2)(z_0).$$

By Lemma 3.2, they are all bounded from above by $-\frac{1}{6}|\mu_\ell|^2(z_0)$, hence non-positive. \square

4. APPLICATION: THE GEOMETRY OF THE WEIL-PETERSSON GEODESIC

An application of our method is to study the geometry of the Weil-Petersson metric in Teichmüller space, in particular, three-manifold formed by a surface bundle over a Weil-Petersson geodesic γ . This manifold is the union of germs N_σ over γ , where $\sigma \in \gamma$, and the fiber at σ is a hyperbolic surface in the conformal class determined by σ . We always assume the curve γ is parametrized by its arc length.

Recall the Hopf differentials $\phi(t)dz^2$ of the family of harmonic maps from (Σ, σ) to $(\Sigma, g_{\rho(t)}dwd\bar{w})$ determine a curve $\rho(t) \in T_g$. It is crucial to us that, when $\phi(t)dz^2$ is a ray in $Q(\sigma)$, i.e., for some $\phi_0 dz^2 \in Q(\sigma) \setminus 0$, $\phi(t) = t\phi_0$, then the slice $\rho(t)$ to second-order is a Weil-Petersson geodesic at $t = 0$ ([Ahl61]). This permits calculation of curvatures and the second fundamental form of the fibers of $N = \bigcup_\sigma N_\sigma$ by local computation on the germs.

This section is organized in subsections. §4.1 contains local calculations where we determine the sectional curvatures of the germ N_σ . The asymptotic behavior of these curvatures near the infinity is also investigated in the subsection; in §4.2, we study the second fundamental form of the fiber hyperbolic surface of N_σ at σ , and show that this fiber is minimal; in §4.3, we equip the surface bundle $N = \bigcup_\sigma N_\sigma$ with a Riemannian structure and prove the Theorem 1.4.

4.1. Curvatures of the germ N_σ . Let γ be a Weil-Petersson geodesic arc in Teichmüller space that passes through (Σ, σ) and in the direction of the harmonic Beltrami differential $\mu_0(z)\frac{d\bar{z}}{dz} = \frac{\bar{\phi}_0 d\bar{z}^2}{g_\sigma dz d\bar{z}} \in B_h(\sigma)$, and $z = x + \sqrt{-1}y$. The Weil-Petersson geodesic arc γ is parametrized by its arc length, so we have

$$\|\mu_0\|_{WP} = \left(\int_{(S, \sigma)} |\mu_0(z)|^2 dA(z)\right)^{\frac{1}{2}} = 1.$$

In this subsection, we focus on local geometry near σ , i.e., we consider the germ N_σ over the point $\sigma \in \gamma$, with the metric H as in (1.8):

$$H = g_{\rho(w)}dwd\bar{w} + dt^2.$$

We obtain sectional curvatures of N_σ with the metric H for tangent vectors $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial t}$, at $t = 0$, and evaluating at $z_0 \in (\Sigma, \sigma)$, that is:

Proof of Theorem 1.3. We set up similarly as in the proof of the Theorem 1.1.

For a family of hyperbolic metrics $g_{\rho(w)}dw d\bar{w}$, we have a family of holomorphic quadratic differentials $\phi(t)dz^2 = t\phi_0dz^2 \in Q(\sigma)$ associated to the harmonic maps $w(t)$ from (Σ, σ) to $(\Sigma, g_{\rho(w)}dw d\bar{w})$.

The pullback metric on (Σ, σ) is given by (3.1):

$$w^*g_{\rho(w)}|dw(t)|^2 = \phi(t)dz^2 + g_\sigma e(t)|dz|^2 + \bar{\phi}(t)d\bar{z}^2,$$

where $e = e(t)$ is again the energy density of $w(t)$, and $\phi(t) = t\phi_0$.

The metric H on the germ N_σ is, in real coordinates, the following:

$$\begin{aligned} & t\phi_0dz^2 + \sigma e(t)dzd\bar{z} + t\bar{\phi}_0d\bar{z}^2 + dt^2 \\ &= t\phi_0(dx^2 - dy^2 + 2idxdy) + g_\sigma e(t)(dx^2 + dy^2) + t\bar{\phi}_0(dx^2 - dy^2 - 2idxdy) + dt^2 \\ &= (g_\sigma e + 2t\text{Re}\phi_0)dx^2 - 4t\text{Im}\phi_0dxdy + (g_\sigma e - 2t\text{Re}\phi_0)dy^2 + dt^2 \end{aligned}$$

The pullback metric on the family of surfaces can be represented by its matrix form as follows:

$$(4.1) \quad (g_{ij}(t)) = \begin{pmatrix} g_\sigma e(t) + 2t\text{Re}\phi_0 & -2t\text{Im}\phi_0 \\ -2t\text{Im}\phi_0 & g_\sigma e(t) - 2t\text{Re}\phi_0 \end{pmatrix}.$$

As before, we use indices 1, 2 for variables x and y , respectively, to simplify the notation in the Christoffel symbols. We now use index 3 for the variable t for the same purpose.

The values of relevant Christoffel symbols, at $t = 0$, can be computed as follows:

$$\begin{cases} \Gamma_{11}^1 = \frac{(g_\sigma)_1}{2g_\sigma}, & \Gamma_{11}^2 = -\frac{(g_\sigma)_2}{2g_\sigma}, & \Gamma_{11}^3 = -\text{Re}\phi_0, \\ \Gamma_{12}^1 = \frac{(g_\sigma)_2}{2\sigma}, & \Gamma_{12}^2 = \frac{(g_\sigma)_1}{2\sigma}, & \Gamma_{12}^3 = \text{Im}\phi_0, \\ \Gamma_{22}^1 = -\frac{(g_\sigma)_1}{2g_\sigma}, & \Gamma_{22}^2 = \frac{(g_\sigma)_2}{2g_\sigma}, & \Gamma_{22}^3 = \text{Re}\phi_0, \end{cases}$$

and

$$\begin{cases} \Gamma_{13}^1 = \frac{\text{Re}\phi_0}{g_\sigma}, & \Gamma_{13}^2 = -\frac{\text{Im}\phi_0}{g_\sigma}, & \Gamma_{13}^3 = 0, \\ \Gamma_{23}^1 = -\frac{\text{Im}\phi_0}{g_\sigma}, & \Gamma_{23}^2 = -\frac{\text{Re}\phi_0}{g_\sigma}, & \Gamma_{23}^3 = 0, \\ \Gamma_{33}^1 = 0, & \Gamma_{33}^2 = 0, & \Gamma_{33}^3 = 0. \end{cases}$$

The curvature tensor R_{1221} , at $t = 0$, is

$$R_{1221}|_0 = g_\sigma(\Gamma_{22,1}^1 - \Gamma_{21,2}^1 + \Gamma_{22}^\beta \Gamma_{\beta 1}^1 - \Gamma_{21}^\beta \Gamma_{\beta 2}^1)|_0.$$

Applying the curvature on (Σ, σ) is -1 , and above values for Christoffel symbols, we have:

$$\begin{aligned} R_{1221}|_0 &= g_\sigma(-g_\sigma + \frac{(\text{Re}\phi_0)^2}{g_\sigma} + \frac{(\text{Im}\phi_0)^2}{g_\sigma}) \\ &= -g_\sigma^2 + |\phi_0|^2. \end{aligned}$$

So we find that, at (σ, z_0) , the curvature in the fiber directions is

$$K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)(\sigma, z_0) = \frac{R_{1221}}{g_\sigma^2} = -1 + |\mu_0(z_0)|^2.$$

We are left to determine curvatures $K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$ and $K\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right)$.

The curvature tensor R_{1331} can be computed as follows:

$$R_{1331}|_0 = g_\sigma(\Gamma_{33,1}^1 - \Gamma_{31,3}^1 + \Gamma_{33}^\beta \Gamma_{\beta 1}^1 - \Gamma_{31}^\beta \Gamma_{\beta 3}^1)|_0,$$

where Einstein notation is employed for $\beta = 1, 2, 3$.

It is easy to verify from the values of the Christoffel symbols at $t = 0$, that

$$\begin{aligned} \sum_{\beta=1}^3 (\Gamma_{33}^\beta \Gamma_{\beta 1}^1 - \Gamma_{31}^\beta \Gamma_{\beta 3}^1)|_0 &= -(\Gamma_{31}^1)^2|_0 - (\Gamma_{31}^2 \Gamma_{23}^1)|_0 \\ &= -|\mu_0|^2, \end{aligned}$$

and we apply the second variation of $e(t)$ to find

$$\begin{aligned} \Gamma_{13,3}^1|_0 &= \frac{1}{2} \left(-\frac{4(Re\phi_0)^2}{g_\sigma^2} + 2D(|\mu_0|^2) + 2|\mu_0|^2 - \frac{4(Im\phi_0)^2}{g_\sigma^2} \right) \\ &= D(|\mu_0|^2) - |\mu_0|^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} R_{1331}|_0 &= g_\sigma(0 - D(|\mu_0|^2) + |\mu_0|^2 - |\mu_0|^2) \\ &= g_\sigma(-D(|\mu_0|^2)), \end{aligned}$$

and at z_0

$$K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)(\sigma, z_0) = \frac{R_{1331}}{g_\sigma} = -D(|\mu_0|^2)(z_0) \leq 0.$$

The last curvature is calculated in the similar fashion and this completes the proof of the Theorem 1.3. \square

We are particularly interested in the case when $\sigma \in T_g$ travels along a Weil-Petersson geodesic γ near the infinity of the augmented Teichmüller space, in which case, at least one simple closed curve on the surface (Σ, σ) is being pinched, as the result, the supremum norm $|\mu|$ is unbounded (more detailed behavior of this supremum norm with respect to the degeneration of the Riemann surface can be found in [Wlp10b]). Therefore, we find:

Corollary 4.1. *Sup $_{\sigma \in T_g} K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = +\infty$, and Inf $_{\sigma \in T_g} K\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) = -\infty$.*

Proof. When $\sigma \in T_g$ is near the infinity of the augmented Teichmüller space, a short essential curve on (Σ, σ) is being pinched, and therefore $Sup_{z \in (\Sigma, \sigma)} |\mu(z)| = +\infty$. Corollary 4.1 is then the consequence of this and the Lemma 3.2. \square

Naturally one hopes that N_σ is negatively curved. Theorem 1.3 gives a negative answer to this. Therefore, as often seen in Riemannian geometry, it is natural to modify a given metric for a better property. On the germ N_σ , based on the metric H in (1.8), we consider the modified metric H_f as follows:

$$(4.2) \quad H_f = g_{\rho(w)} dw d\bar{w} + f(t) dt^2,$$

where $f(t) > 0$ and $f(0) = 1$. Proceeding as in the proof of Theorem 1.3, it is easy to show the following:

Proposition 4.2. *The curvature $K(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ of the Weil-Petersson geodesic γ in the fiber directions, with respect to the metric H_f , at $t = 0$, and $z = z_0$ is equal to $-1 + |\mu_0|^2(z_0)$. Therefore (N_σ, H_f) is not negatively curved.*

4.2. Minimality: the germ N_σ . We now consider how the fiber hyperbolic surface $(\Sigma, g_\sigma|dz|^2)$ interacts with the germ N_σ . Naturally we study the second fundamental form.

Since the t -direction is perpendicular to each surface, the second fundamental form $(h_{ij}(t))$ can be calculated as

$$(h_{ij}(t)) = \begin{pmatrix} g_\sigma e'(t) + 2\operatorname{Re}\phi_0 & -2\operatorname{Im}\phi_0 \\ -2\operatorname{Im}\phi_0 & g_\sigma e'(t) - 2\operatorname{Re}\phi_0 \end{pmatrix}.$$

Here $e'(t)$ is the t -derivative of the energy density $e(t)$ of the family of harmonic maps $w(t)$.

Since $e(0) = 1$ and $e'(0) = 0$, when evaluated at $t = 0$, we have

$$(4.3) \quad (h_{ij}(0)) = \begin{pmatrix} 2\operatorname{Re}\phi_0 & -2\operatorname{Im}\phi_0 \\ -2\operatorname{Im}\phi_0 & -2\operatorname{Re}\phi_0 \end{pmatrix}.$$

Now the principal curvatures of surface at time $t = 0$ are the eigenvalues of the following matrix:

$$(g_\sigma)^{-1}(h_{ij}(0)) = \frac{1}{g_\sigma} \begin{pmatrix} 2\operatorname{Re}\phi_0 & -2\operatorname{Im}\phi_0 \\ -2\operatorname{Im}\phi_0 & -2\operatorname{Re}\phi_0 \end{pmatrix}.$$

Its two eigenvalues are now:

$$(4.4) \quad \lambda = \pm 2 \frac{|\phi_0|}{g_\sigma} = \pm 2|\mu_0|.$$

Therefore we have proved:

Theorem 4.3. *Each fiber hyperbolic surface $(S, g_\sigma dzd\bar{z})$ is a minimal surface of the germ N_σ .*

We note that the principal curvatures are unbounded if σ is near the infinity of the augmented Teichmüller space.

4.3. Minimality: surface bundle N over γ . Let $\gamma(t)$ be a Weil-Petersson geodesic arc, for $0 \leq t \leq T$, parametrized by its arc length. We denote the collection of germs N_σ for $\sigma \in \gamma(t)$ by N , and it is clear that the three-manifold $N = \Sigma \times [0, T]$ is a surface bundle over $\gamma(t)$. In this subsection, we prove the Theorem 1.4.

proof of Theorem 1.4. We now equip N with a Riemannian structure. We wish to equip N with a metric of the form

$$(4.5) \quad g_{\gamma(t)} dw(t) d\bar{w}(t) + dt^2$$

where $g_{\gamma(t)}$ is the hyperbolic metric in the conformal class of $\gamma(t)$.

Let \mathcal{M}_{-1} be the space of hyperbolic metrics on a topological surface S , and recall that Teichmüller space is the space of hyperbolic metrics on S , up to orientation-preserving diffeomorphisms in the homotopy class of the identity: $T_g = \mathcal{M}_{-1}/D_0$, where D_0 is the identity component of the diffeomorphism group.

Let $[g_{\gamma(t)}]$ denote the fiber over $g_{\gamma(t)} \in T_g$. At any $g_1 \in [g_{\gamma(t)}]$, the tangent space $T_{g_1} \mathcal{M}_{-1}$ is identified as $\operatorname{Sym}(0, 2)$, the space of symmetric $(0, 2)$ -tensors on S . Let $h_1 \in \operatorname{Sym}(0, 2)$ be the symmetric $(0, 2)$ -tensor which induces a deformation of g_1 preserving the scalar curvature. It is divergence-free and traceless, and moreover

the deformation is smoothly dependent in t by Theorem A of [FM75], or Theorems 2.4.2 of [Tro92]. In our case, g_1 is the hyperbolic metric $\sigma \in [g_{\gamma(t)}]$ and h_1 is the holomorphic quadratic differential in the tangent direction $\phi_0 dz^2$, and the smooth dependence results in a C^∞ Riemannian metric on N such that the germs at each fiber is identical to N_σ .

Note that the metric (4.5) agrees with metric (1.8) associated to $\rho(t)$ up to second order ([Ahl61]), and now parts (1) and (2) follow from Theorem 4.3, and part (3) follows from the Theorem 1.3. \square

By a theorem of Sullivan ([Sul79]), any compact Riemannian manifold with a taut foliation admits a Riemannian metric such that each leaf of the foliation is a minimal surface. Minimal foliations lead to many interesting results in both minimal surface theory and the topology of closed 3-manifolds (see [Has05]).

Corollary 4.4. *If γ is a closed Weil-Petersson geodesic loop in moduli space $\mathcal{M}(S)$, then the metric (4.5) on the associated 3-manifold N is a Sullivan metric.*

Proof. The Weil-Petersson geodesic loop lifts to a path of hyperbolic metrics $\gamma(t)$ in \mathcal{M}_{-1} , as in the proof of Theorem 1.4, such that the hyperbolic surfaces $\Sigma_0 = \Sigma \times \{0\}$ and $\Sigma_1 = \Sigma \times \{T\}$ are isometric by an orientation-preserving mapping class $\psi : \Sigma_0 \rightarrow \Sigma_1$. We glue the boundary of $\Sigma \times [0, T]$ equipped with the metric (4.5) by ψ to obtain a surface bundle N . Since the normal direction is $\partial/\partial t$, the normal vectors $\partial/\partial t$ on Σ_0 match up with the normal vectors on Σ_1 , and the resulting metric on N is smooth. The property of all the fibers being minimal follows from Theorem 1.4. \square

Remark 4.5.

- (1) *The minimality of the germs does not require $\gamma \subset T_g$ to be a geodesic. It is essentially due to the property that deformations h_1 can be chosen as traceless. Being a Weil-Petersson geodesic allows us to apply the method of harmonic maps to calculate the curvatures of N .*
- (2) *Each fiber of N has the same surface area. It is possible to provide an expression for the associated calibration ω , a 2-form of co-mass 1, on N . It is the hyperbolic area form when it is restricted on each fiber and it is closed since the first variation of the area along a Weil-Petersson geodesic vanishes ([Ahl61],[Wlp86]).*

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