BIFURCATION FOR MINIMAL SURFACE EQUATION IN HYPERBOLIC 3-MANIFOLDS

ZHENG HUANG, MARCELLO LUCIA, AND GABRIELLA TARANTELLO

Abstract. Initiated by the work of Uhlenbeck in late 1970s, we study existence, multiplicity and asymptotic behavior for minimal immersions of a closed surface in some hyperbolic three-manifold, with prescribed conformal structure on the surface and second fundamental form of the immersion. We prove several results in these directions, by analyzing the Gauss equation governing the immersion. We determine when existence holds, and obtain unique stable solutions for area minimizing immersions. Furthermore, we find exactly when other (unstable) solutions exist and study how they blow-up. We prove our class of unstable solutions exhibit different blow-up behaviors when the surface is of genus two or greater. We establish similar results for the blow-up behavior of any general family of unstable solutions. This information allows us to consider similar minimal immersion problems when the total extrinsic curvature is also prescribed.

0. Introduction: Geometric Settings

Minimal surfaces have long been a fundamental object of intense study in geometry and analysis. In this paper we study minimal immersions of a closed surface in some hyperbolic three-manifolds. Inspired by Uhlenbeck's approach ([**Uhl83**]), results on existence and multiplicity of such minimal immersions, as well as their geometrical interpretations, are obtained by analyzing bifurcation properties of solutions to the minimal surface equation. Throughout the paper, we assume S is a closed oriented surface of genus $g \geq 2$. The Teichmüller space of S is denoted by $T_g(S)$, and it is the space of conformal structures (or equivalently hyperbolic metrics) on S such that two conformal structures are equivalent if there is between them an orientation-preserving diffeomorphism in the homotopy class of the identity.

When S is immersed in some hyperbolic three-manifold M, we denote by g_0 the induced metric from the immersion and by $\sigma \in T_g(S)$ the conformal structure on S induced by g_0. Furthermore, we denote by g_σ the unique hyperbolic metric on (S, σ), and by dA its area form. Since the metrics g_σ and g_0 are conformally equivalent, for a suitable conformal factor $u \in C^\infty(S)$, we have

$$g_0 = e^{2u} g_\sigma. \tag{0.1}$$

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We denote always by $z = x + iy$ the conformal coordinates on (S, σ). So in local conformal coordinates we may write:

$$g_\sigma = e^{2u_\sigma} dz d\bar{z}, \quad \text{and} \quad g_0 = e^{2v} dz d\bar{z},$$

with $v = u_\sigma + u$ and u given in (0.1). Now, in such coordinates, the second fundamental form II takes the following quadratic expression:

$$\mathrm{II} = h_{11}(dx)^2 + 2h_{12}dxdy + h_{22}(dy)^2, \tag{0.2}$$

with $h_{11} = -h_{22}$ accounting for the fact that (S, g_0) is a minimal surface in M.

The Riemann curvature tensor $R_{ij\kappa\ell}$ and the metric tensor $g = (g_{ij})$ of the hyperbolic three-manifold (M, g) satisfy the following equations:

$$R_{ij\kappa\ell} = -(g_{i\kappa}g_{j\ell} - g_{i\ell}g_{j\kappa}). \tag{0.3}$$

Note that by Bianchi identities, only six components of $R_{ij\kappa\ell}$ are independent.

In this respect, we can use normal coordinates (z, r) for the normal bundle $T^N(S)$, with conformal coordinates $z \in S$ and $r \in (-a, a)$ for some $a > 0$ small. We obtain via the exponential map a local coordinate system on M around S, where we have $g_{j3} = \delta_{j3}$, for $j = 1, 2, 3$. In these coordinates, the remaining components $g_{i\kappa}$, $1 \le i \le \kappa \le 2$, are just $-R_{i3\kappa3}$ in view of (0.3), namely,

$$R_{i3\kappa3} = -g_{i\kappa}. \tag{0.4}$$

The equations in (0.4) can be viewed as a second order system of ODEs for g_{ik} in the variable r (and fixed $z \in S$). So we can uniquely identify $g_{i\kappa}$ (around S) by its initial data:

$$\begin{cases} g_{i\kappa}(z, 0) = (g_0)_{i\kappa}(z) \\ \frac{1}{2}\frac{\partial}{\partial r} g_{i\kappa}(z, 0) = h_{i\kappa}(z), \quad 1 \le i \le \kappa \le 2. \end{cases} \tag{0.5}$$

Such initial data on S are provided by the remaining equations in (0.3). To verify this, we take $\ell = 3$ and $j \ne 3$ in (0.3) and get

$$R_{ij\kappa3} = 0, \tag{0.6}$$

which expresses the Codazzi equations on S, for $1 \le i, j, \kappa \le 2$. Again only two of those equations are independent, and they ensure exactly that the quadratic differential $\alpha = (h_{11} - ih_{12})dz^2$ is holomorphic and

$$\mathrm{II} = Re(\alpha), \tag{0.7}$$

see [[**Hop89, LJ70**]]. In other words, $\alpha \in Q(\sigma)$, where we denote by $Q(\sigma)$ the space of holomorphic quadratic differentials on (S, σ).

Finally taking $i = \kappa = 1$ and $j = \ell = 2$ in (0.3), one gets:

$$R_{1212} = -g_{11}g_{22} + g_{12}^2, \tag{0.8}$$

which simply gives the Gauss equation on S, and it states that the conformal factor $u(z)$ in (0.1) on S must satisfy:

$$\Delta u + 1 - e^{2u} - \frac{|\alpha|^2}{det(g_\sigma)} e^{-2u} = 0, \tag{0.9}$$

with α given in (0.7), and Δ the Laplacian in the hyperbolic metric g_σ. Indeed, (0.9) simply expresses a consistency condition on (S, g_σ) between the intrinsic curvature of the metric g_0 and the extrinsic curvature $det_{g_0}\mathrm{II}$, see [**Uhl83**] for details.

Note that, by Bianchi identities, once the equations (0.6), (0.8) hold on S then they hold throughout the normal bundle of S. Furthermore, these equations on S provide the initial data (0.5) in terms of (σ, α), simply by using the solutions of the Codazzi-Gauss equations (0.7), (0.9) into (0.1)-(0.2).

Thus by prescribing $\sigma \in T_g(S)$ and $\alpha \in Q(\sigma)$ such that the Codazzi-Gauss equations (0.7), (0.9) are solvable, it is natural to ask whether it is possible to obtain a minimal immersion of (S, σ) into some hyperbolic three-manifold with the second fundamental form satisfying (0.7). In short, we shall call a *minimal immersion of S with data* (σ, α) any of such minimal immersion.

A general construction of a minimal immersion with prescribed data satisfying the Codazzi-Gauss equations (called "hyperbolic germs" in [**Tau04**]) is available in literature, see for instance [**Tau04, Jac82**]. However, it is not always possible to guarantee that the corresponding hyperbolic three-manifold is complete, unless we are more specific about the induced metric g_0 or equivalently about the solution of (0.9). Thus, to obtain more satisfactory results of geometrical nature, Uhlenbeck analyzed in [**Uhl83**] more closely the set of solutions of (0.9).

We recall that a solution u of (0.9) is called *stable* if the linearized operator of (0.9) at u is nonnegative definite in $H^1(S)$, and called *strictly stable* if the linearized operator of (0.9) at u is positive definite in $H^1(S)$. The interest to stable solutions is justified by the fact that they give rise to (local) area minimizing immersions.

By setting:

$$\|\alpha\|_\sigma^2 = \frac{|\alpha|^2}{det(g_\sigma)} = \frac{1}{2}\|\mathrm{II}\|_{g_\sigma}^2,$$

the length squared of α with respect to the hyperbolic metric g_σ, Uhlenbeck ([**Uhl83**]) considered a one-parameter family of Gauss equations:

$$\Delta u + 1 - e^{2u} - t^2\|\alpha\|_\sigma^2 e^{-2u} = 0, \tag{0.10}$$

for minimal immersions of S with data $(\sigma, t\alpha)$. Using the implicit function theorem, she proved the existence and uniqueness of a smooth solution curve of stable solutions to the equation (0.10):

Theorem 0.1. *(*[**Uhl83**]*) Fixing a conformal structure $\sigma \in T_g(S)$, and $\alpha \in Q(\sigma)$, there exists a constant $\tau_0 > 0$, depending only on (σ, α), such that the equation* (0.10) *admits a unique stable solution if and only if $t \in [0, \tau_0]$. Furthermore for each $t \in [0, \tau_0]$, the stable solution $u_t < 0$ of* (0.10) *forms a smooth monotone decreasing curve with respect to t. Moreover, u_t is strictly stable for $t \in [0, \tau_0)$ and $u_t \nearrow u_{t=0} = 0$, as $t \searrow 0$, in $H^1(S)$.*

From Theorem 0.1, a bifurcation diagram (especially for the lower branch, in absolute value, of the solution curve) can be sketched as below in Figure 1:

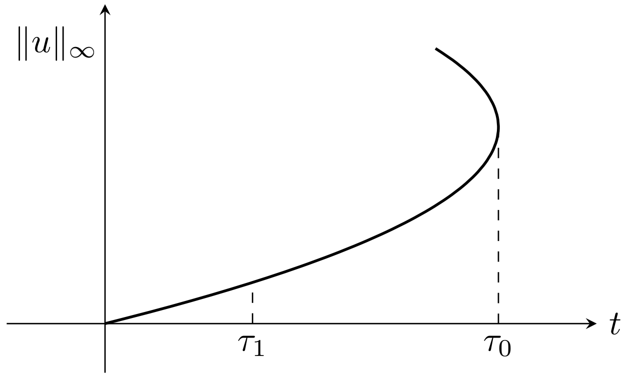

FIGURE 1. Uhlenbeck's Solution Curve

In this diagram, Uhlenbeck indicated a first turn of the curve of stable solutions at some τ_0, though it is still possible the curve retracts and passes again the value $t = \tau_0$ to join other solutions of (0.10) for $t \geq \tau_0$. Actually it is our first task here to show that this is never the case.

From the geometrical point of view, by Theorem 0.1 we know that, for $t \in [0, \tau_0]$, there exists an area minimizing (stable) immersion of S with data $(\sigma, t\alpha)$ whose induced metric on the surface (S, σ) is $g_0 = e^{2u_t} g_\sigma$. The second variation of the area functional is explicitly computed in terms of the linearized operator in [**Uhl83**]. Also it is interesting to note that there exists a $\tau_1 > 0$, such that for each $t \in (0, \tau_1)$, the surface can be minimally immersed into a so-called almost Fuchsian manifold and this almost Fuchsian manifold contains (S, σ) as its unique minimal surface. In particular, as $t \to 0^+$ such family of almost Fuchsian hyperbolic three-manifolds converge (in the sense of Gromov-Hausdorff) to a Fuchsian manifold where S is embedded as a totally geodesic area minimizing surface, see [**Uhl83**] for details.

Further work in [**HL12**] obtained an additional solution for each of Uhlenbeck's (strictly) stable solution to the Gauss equation:

Theorem 0.2. *(*[**HL12**]*) Let S be a closed surface and $\sigma \in T_g(S)$ be a conformal structure on S. If $\alpha \in Q(\sigma)$ is a holomorphic quadratic differential on (S, σ), then:*

- **i)** *for sufficiently large t, the Gauss equation* (0.10) *admits no solutions, i.e., there is no minimal immersion of S with data $(\sigma, t\alpha)$ into any hyperbolic three-manifold;*
- **ii)** *for each $t \in (0, \tau_0)$, with $\tau_0 > 0$ given in Theorem 0.1, there exists also an unstable immersion of S with data $(\sigma, t\alpha)$.*

These results reveal further details on the solution curve to (0.10) and an improved bifurcation diagram can be sketched as follows:

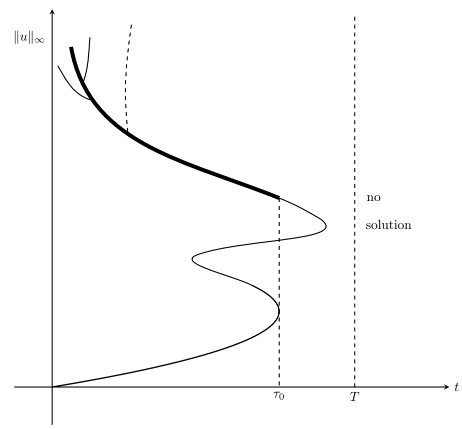

FIGURE 2. Solution Curve from [**HL12**]

1. Introduction: Main results

The first purpose of this paper is to complete the above results in Theorems 0.1 and 0.2 as follows. Firstly, we already mentioned, we show that actually the interval $[0, \tau_0]$ exhausts the full range of values $t \geq 0$ for which the equation (0.10) is solvable. Namely, the bifurcation curve starting from the trivial solution at $t = 0$, cannot admit an "S-shape" (as typical in similar problems), but it turns only at τ_0 and until then, furnishes the lower branch (in absolute value) of unique (strictly) stable solutions of (0.10). In fact, equation (0.10) admits a unique (stable but not strictly stable) solution for $t = \tau_0$ and no solutions for $t > \tau_0$. Furthermore we provide a family of unstable solutions for (0.10) with a specific asymptotic behavior, as $t \to 0^+$. A sketch of the bifurcation diagram can be seen as below in Figure 3.

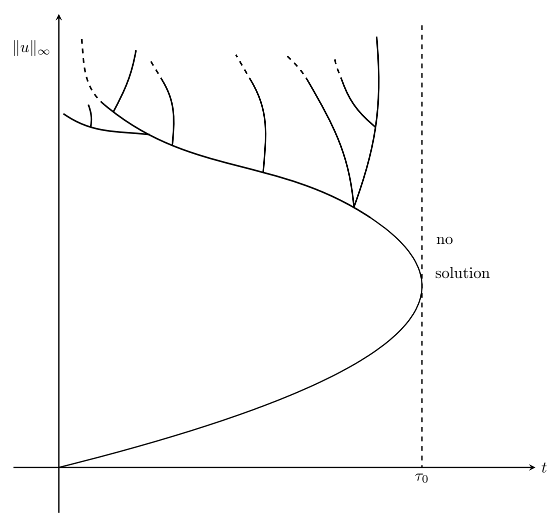

FIGURE 3. New Solution Curve

Theorem A. *Fixing a conformal structure $\sigma \in T_g(S)$, and a holomorphic quadratic differential $\alpha \in Q(\sigma)$, the equation* (0.10) *admits a solution if and only if $t \in [0, \tau_0]$, with $\tau_0 = \tau_0(\sigma, \alpha) > 0$ given in Theorem 0.1. Furthermore, the unique stable solution $u_t < 0$ is the (pointwise) largest solutions of* (0.10) *for $t \in [0, \tau_0]$. Moreover,*

- **(i)** *$\forall t \in (0, \tau_0)$, the equation* (0.10) *admits an <u>unstable</u> solution $\tilde{u}_t$ (with $\tilde{u}_t < u_t < 0$ on S) such that, as $t \searrow 0$,*
$$\max_S |\tilde{u}_t| \to +\infty,$$
- **(ii)** *for $t = \tau_0$, the equation* (0.10) *admits the unique solution u_0:*
$$u_0(z) = \lim_{t \nearrow \tau_0} u_t(z) = \inf_{t \in (0, \tau_0)} u_t(z), \ \forall z \in S$$

We may say that (τ_0, u_{τ_0}) is a "bending point" for the bifurcation curve starting at $(t = 0, u = 0)$, using the terminology introduced in [**AR73**]. Actually we provide a much more detailed study of the asymptotic behavior of the unstable solution $\tilde{u}_t$ in Theorem A, as $t \searrow 0$. Interestingly, its behavior depends on whether the surface (S, σ) can be "uniformized" within the class of hyperbolic metrics with conical singularities along a divisor of modulus 2.

To be more precise, recall that a divisor D on S is a "formal" expression of the type

$$D = \sum_{j=1}^{m} \alpha_j p_j, \tag{1.1}$$

with given points $p_j \in S$ and values $\alpha_j > -1, \ j = 1, \cdots, m$; and

$$|D| = \sum_{j=1}^{m} \alpha_j$$

is the modulus of the divisor. Therefore a hyperbolic metric on S with conical singularities along the divisor D in (1.1) is a metric with Gauss curvature -1 in $S \backslash \{p_1, \cdots, p_m\}$ and a conical singularity at p_j with angle $\theta_j = 2\pi(1 + \alpha_j)$ for $j = 1, \cdots, m$. Thus, the "uniformization" of (S, σ) in the class of hyperbolic metrics with conical singularities along the divisor D, just means that at least one of such metric is conformal to g_σ. Consequently, the corresponding conformal factor u is given by the <u>unique</u> solution of the following equation:

$$-\Delta u = 1 - e^{2u} - 2\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j}. \tag{1.2}$$

By analyzing (1.2), one easily see that the "uniformization" of S in the sense above is possible if and only if

$$\chi(S) + |D| < 0,$$

where $\chi(S) = 2(1-g)$ is the Euler characteristic of S. In particular, if $|D| = 2$, then necessarily the genus $g \geq 3$. Thus our blow-up analysis must distinguish between the cases where S has genus two or higher.

We start with the following:

Theorem B. *Let the genus of the surface S satisfy $g \geq 3$, and $\tilde{u}_t$ be the unstable solution given by Theorem A. Then as $t \searrow 0$, we have:*

$$t^2 \|\alpha\|_\sigma^2 e^{-2\tilde{u}_t} \rightharpoonup 4\pi\delta_{p_0} \tag{1.3}$$

weakly in the sense of measures, with some point $p_0 \in S$ such that $\alpha(p_0) \neq 0$. Furthermore:

$$\tilde{u}_t \to \tilde{u} \quad \text{in } W^{1,q}(S), 1 < q < 2, \quad \text{uniformly in } C^{2,\beta}_{loc}(S \backslash \{p_0\}), 0 < \beta < 1,$$

and

$$e^{2\tilde{u}_t} \to e^{2\tilde{u}} \qquad \text{in } L^s(S), \quad \forall s \geq 1,$$

with $\tilde{u}$ the unique solution of the following problem on S.:

$$\Delta\tilde{u} + 1 - e^{2\tilde{u}} - 4\pi\delta_{p_0} = 0. \tag{1.4}$$

From the geometrical point of view, Theorem B states that, $\forall t \in (0, \tau_0)$ there exist (unstable) minimal immersions of S with data $(\sigma, t\alpha)$ which converge (in the sense of Gromov-Hausdorff) as $t \to 0^+$ to a "limiting" totally geodesic immersion of S into a three-dimensional hyperbolic cone-manifold of the type introduced by Krasnov-Schlenker [**KS07**]. More precisely, the given hyperbolic three-manifold contains conical singularities along one line to infinity and the induced metric on S is hyperbolic with a conical singularity along the divisor $D = 2p_0$, with suitable $p_0 \in S$ and $\alpha(p_0) \neq 0$. Therefore for our unstable immersions, we observe a quite different limiting behavior from the case of area minimizing (stable) immersions.

If S has genus $g = 2$, then the asymptotic behavior of $\tilde{u}_t$ is described by a "concentration-compactness" alternative, resolved by the existence (or not) of a minimum for a Moser-Trudinger type functional. To be more precise, let us recall the following:

Definition 1.1. *Let $E = \{w \in H^1(S) \text{ with } \int_S w(z)dA = 0\}$, the Moser-Trudinger functional on (S, σ) with weight function $0 \leq K \in L^\infty(S)$ is given by*

$$\mathcal{J}(w) = \frac{1}{2}\int_S |\nabla w|^2 \; dA - 8\pi \log(\fint_S K(z)e^w dA), \quad w \in E, \tag{1.5}$$

where we have used the standard notation:

$$\fint_S f \; dA = \frac{\int_S f \; dA}{|S|},$$

and the area $|S| = 4\pi(g-1)$ by the Gauss-Bonnet theorem.

By the Moser-Trudinger inequality (see [**Aub98**]), the functional $\mathcal{J}$ is bounded from below but <u>not</u> coercive in E. In other words, it is well defined

$$\inf_E \mathcal{J} > -\infty, \tag{1.6}$$

but the infimum in (1.6) may not be attained.

For the case of genus $g = 2$, our main result reads:

Theorem C. *If the surface S is of genus $g = 2$, and $\tilde{u}_t$ is the unstable solution given by Theorem A, then, as $t \searrow 0$,*

$$\int_S |\tilde{u}_t| \; dA \to +\infty.$$

Furthermore, for $K(z) = \|\alpha\|_\sigma^2$, we have the following alternatives:

(i) *(Compactness) <u>either</u>, the Moser-Trudinger functional $\mathcal{J}$ (with $K = \|\alpha\|_\sigma^2$) attains its infimum in E, and along a sequence $t_n \to 0$, there holds:*

$$(\tilde{u}_{t_n} - \fint_S \tilde{u}_{t_n}) \to \hat{w}_0, \qquad \text{strongly in } \; H^1(S),$$

and,

$$t_n^2 K(z) e^{-2\tilde{u}_{t_n}} \to \frac{4\pi K(z) e^{-2\hat{w}_0}}{\int_S K(z) e^{-2\hat{w}_0} \; dA} \qquad \text{uniformly in } \; C^{2,\beta}(S),$$

with $\hat{w}_0$ satisfying on (S, σ),

$$\begin{cases} \Delta \hat{w}_0 + 4\pi \left(\frac{1}{|S|} - \frac{K(z) e^{-2\hat{w}_0}}{\int_S K(z) e^{-2\hat{w}_0} dA} \right) = 0 \\ \mathcal{J}(-2\hat{w}_0) = \inf\limits_E \mathcal{J}; \end{cases} \tag{1.7}$$

(ii) *(Concentration) <u>or</u>, the functional $\mathcal{J}$ (with $K = \|\alpha\|_\sigma^2$) does not attain its infimum in E, and along a sequence $t_n \to 0$, there holds:*

$$t_n^2 K(z) e^{-2\tilde{u}_{t_n}} \rightharpoonup 4\pi \delta_{p_0}, \tag{1.8}$$

weakly in the sense of measure, for some $p_0 \in S$ such that $\alpha(p_0) \neq 0$,

$$(\tilde{u}_{t_n} - \fint_S \tilde{u}_{t_n}) \to 4\pi G(\cdot, p_0), \quad \text{in } \; W^{1,q}(S), 1 < q < 2, \tag{1.9}$$

and uniformly in $C^{2,\beta}_{loc}(S \backslash \{p_0\}), 0 < \beta < 1$, with $G(\cdot, p)$ the unique Green's function of the Laplace operator Δ on the hyperbolic surface (S, g_σ), as defined in (3.1) *below.*

Hence, from Theorem C we see that, in contrast to higher genera, when $g = 2$ the (unstable) minimal immersion of S does not survive the passage to the limit, as $t \to 0^+$. Indeed for the induced metric $\tilde{g}_t^0 = e^{2\tilde{u}_t} g_\sigma$, there holds:

$$|S|_{\tilde{g}_t^0} \to 0, \text{ as } t \to 0^+.$$

Nonetheless, when the functional $\mathcal{J}$ in (1.5) with $K = \|\alpha\|_\sigma^2$ attains its minimum in E, then we can still find, as $t \to 0^+$, a "limiting" configuration for the "blown-up" surface $(S, \hat{g}_t^0)$ with

$$\hat{g}_t^0 = e^{2\hat{u}_t} g_\sigma, \text{ and } \quad \hat{u}_t = \tilde{u}_t - \log t.$$

Indeed, along a sequence $t = t_n \to 0^+$, the blown-up sequence $(S, \hat{g}_{t_n}^0)$ converges (in the sense of Gromov-Hausdorff) to a surface $(S, \hat{g}^0)$ conformally equivalent to (S, g_σ) with same total negative curvature -4π. It is interesting to note that $(S, \hat{g}^0)$ admits a non-positive curvature vanishing exactly as $|\alpha|$.

More generally, for any sequence of unstable solutions u_n of (0.10) with $t = t_n \to 0^+$, as $n \to +\infty$, we shall carry out an analogous blow-up analysis whose details are contained in Theorem D of Section §3. Roughly speaking, to any such sequence, we shall associate a divisor D in S of the following type:

$$D = 2\sum_{j=1}^{m}(1 + n(p_j))\delta_{p_j} \tag{1.10}$$

with suitable $p_j \in S$ and $n(p_j) = 0$ if $\alpha(p_j) \neq 0$, while for p_j a zero of α (i.e. $\alpha(p_j) = 0$), then $n(p_j)$ is given by the corresponding multiplicity. Furthermore,

$$\chi(S) + |D| \leq 0. \tag{1.11}$$

In case $\chi(S) + |D| < 0$, then as before, we find that the sequence of corresponding minimal immersions of S with data $(\sigma, t_n\alpha)$ converges (in the sense of Gromov-Hausdorff) to a "limiting" totally geodesic immersion into a three-dimensional hyperbolic cone-manifold which induces on S a hyperbolic metric with conical singularities along the divisor D in (1.10). While in case $\chi(S) + |D| = 0$, then no such singular metric exist on S and exactly as above, the given (unstable) minimal immersions do not survive the passage to the limit. Thus, as before, the asymptotic behavior of the blow-up sequence u_n is described by a "concentration-compactness" alternative, see Theorem D in section §3 for details.

It is an interesting open problem to see whether unstable solutions of (0.10) can be constructed in such a way that their blow-up behavior matches a prescribed divisor of the type (1.10) and (1.11).

In conclusion, let us make a few remarks.

Remark 1.2. *It is well known that any $\alpha \in Q(\sigma)$ admits $4(g-1)$ zeroes, counting multiplicity. Seen from Theorems B and C, we have that the blow-up of the unstable solution $\tilde{u}_t$, as $t \to 0$, cannot occur around a zero of α. A more precise characterization of the blow-up point p_0 in Theorems B and C will be given in the sections §4 and §5.*

Remark 1.3. *It is interesting to record that for $g = 2$, the behavior of the unstable solution $\tilde{u}_t$, as $t \to 0^+$ depends on whether the Moser-Trudinger functional $\mathcal{J}$ in* (1.5) *with weight $K = \|\alpha\|_\sigma^2$ attains in its infimum in E. Actually, exactly when $K(z) = \|\alpha\|_\sigma^2$, the existence of extrema for $\mathcal{J}$ appears to be a delicate open problem. Indeed we shall see in section §5.2 that in such case the functional $\mathcal{J}$ just misses to satisfy the condition provided (for a general weight function K) in Theorem 7.2 of* [**DJLW97**]*] which is sufficient to ensure the existence of a global minimum for $\mathcal{J}$.*

Remark 1.4. *From the blow-up analysis in Theorem D where Theorems B and C enter as special cases, we shall obtain a compactness result for solutions of* (0.10)*. This will enable us to obtain a minimal immersion of S with prescribed total extrinsic curvature, $\rho = \int_S det_{g_0}(II)dA(g_0) \in (0, 4\pi(g-1))$ and data $(\sigma, t_\rho\alpha)$, with suitable $t_\rho \in (0, \tau_0)$. It would be interesting to investigate the dependence of t_ρ with respect to ρ.*

Remark 1.5. *(Added to the proof) The statement of Theorem A and parts of its proof are somewhat similar to a result of Ding-Liu (*[**DL95**]*) where they analyzed a problem of prescribing Gaussian curvature on closed surfaces. In fact from the* *variational point of view, both problems (with a parameter) admit similar structures. More precisely, we have the presence of a (strict) local minimum and a "mountain pass geometry", for a suitable sharp range of the parameter involved. Typically, in this situation one can claim the existence of a stable and unstable solution (as in Theorem A and in* [**DL95**]*) with the stable solution "bifurcating" out of a known (trivial) solution of the problem at a limiting value of the parameter. Much more interesting is the description of the asymptotic behavior of the unstable solution, which reflects the particular nature of the geometrical problem in hands. This is the purpose of our Theorems B, C and D. For the Gauss curvature problem treated in* [**DL95**]*, this goal has been pursued in* [**BGS15**]*, and more recently by Struwe in* [**Str20**]*, who obtained complete results. In fact, Struwe's result encourages the possibility to also complete our blow-up analysis in Theorem D when blow-up occurs at a zero of the quadratic differential α, see Remark 6.3 and 6.4 below. We thank the referee for pointing out these references.*

Plan of the rest of the paper: In §2, we will provide several estimates before we move to prove Theorem A in section §4. Detailed blow-up analysis is conducted in sections §3 where we obtain Theorem D, and in §5, where we prove Theorems B and C. In §6, we extend the program to explore this minimal immersion problem when prescribing the total extrinsic curvature.

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2. Elementary estimates

Before we proceed, we introduce more convenient notations. We set

$$v = -2u$$

and

$$K(z) = \|\alpha\|_\sigma^2 = \frac{|\alpha|^2}{det(g_\sigma)},$$

and rewrite the Gauss equation (0.10) as follows:

$$-\Delta v = 2t^2 K e^v - 2(1 - e^{-v}), \tag{2.1}$$

where $v \in H^1(S)$, $t \geq 0$, and $K(z) \geq 0$ has finitely many zeroes, given by the zeroes of the prescribed holomorphic quadratic differential $\alpha \in Q(\sigma)$, whose total number is $4g - 4$, counting multiplicity.

Definition 2.1. *We call a function $v_t \in H^1(S)$ a solution of problem $(1)_t$ for $t \geq 0$, if it solves the equation* (2.1).

We collect some basic properties for the solutions of this problem.

Lemma 2.2. *If v is a solution of problem $(1)_t$, then we have*

i)

$$t^2 \int_S K(z)e^v dA + \int_S e^{-v} dA = 4\pi(g-1), \tag{2.2}$$

and in particular,

$$(2\pi(g-1))^2 \geq t^2 \int_S K(z)e^v dA \int_S e^{-v} dA, \tag{2.3}$$

ii) *$v \geq 0$ and $v \equiv 0$ if and only if $t = 0$. Therefore $v(z) > 0$ for all $z \in S$ for any $t > 0$.*

iii) *If we write $v = w + \mathsf{c}$, with $\int_S w(z)dA = 0$, and $\mathsf{c} = ⨍_S v(z)dA$, then $\mathsf{c} > 0$ and*

$$e^{\mathsf{c}} = \frac{2\pi(g-1) \pm \sqrt{(2\pi(g-1))^2 - t^2 \int_S K(z)e^w dA \int_S e^{-w} dA}}{t^2 \int_S K(z)e^w dA}. \tag{2.4}$$

Proof. These properties follow by direct and elementary calculations. More specifically, to obtain (2.2), we integrate (2.1) and apply the Gauss-Bonnet formula. At this point (2.3) is a direct consequence of (2.2) and Schwarz inequality.

In order to show (ii), we simply write $v = v^+ - v^-$ where v^+ and v^- are both non-negative. Using v^- as a test function we have

$$\int_S \nabla v \nabla v^- \; dA = 2\left(\int_S t^2 K e^v v^- + (e^{-v} - 1)v^- \; dA\right),$$

and this is equivalent to

$$-\int_S |\nabla v^-|^2 \; dA = 2\left(\int_S t^2 K e^v v^- + (e^{v^-} - 1)v^- \; dA\right),$$

Since $v^- \geq 0$, we find

$$\int_S (e^{v^-} - 1)v^- \; dA \leq 0.$$

But the scalar function $(e^x - 1)x \geq 0$ whenever $x \geq 0$, we deduce that $v^- \equiv 0$ and therefore $v \geq 0$. Furthermore, since $v \in H^1(S)$, the righthand side of (2.3) is in L^p for any $p \geq 1$. Standard regularity results for elliptic equations imply that v is in fact smooth. Then the conclusion follows from the strong maximum principle.

Finally (iii) follows by using $v = w + \mathsf{c}$ in (2.2) and by solving with respect to c. □

For later use and according to the choice of the sign in (2.4), we set:

$$\mathsf{c}_\pm = \log\{\frac{2\pi(g-1) \pm \sqrt{(2\pi(g-1))^2 - t^2 \int_S K(z)e^w dA \int_S e^{-w} dA}}{t^2 \int_S K(z)e^w dA}\}. \tag{2.5}$$

Lemma 2.3. *(**[HL12]**) If $(1)_t$ admits a solution, then*

$$t \leq \frac{1}{⨍_S 2\sqrt{K(z)}dA} = t^*. \tag{2.6}$$

Proof. Inequality (2.6) easily follows from (2.2) and the Cauchy-Schwarz inequality, as seen in [**HL12**]. □

In particular, this lemma implies that, for $t > t^*$, there does not exist any minimal immersion of S with data $(\sigma, t\alpha)$.

Lemma 2.4. *Let $v = w + \mathsf{c}$ be a solution to problem $(1)_t$, where w and c are as in Lemma 2.2. We have*

i) *For any $s \in [1, 2)$, there exists a constant $C_s > 0$ such that*

$$\|\nabla w\|_{L^s} \leq C_s. \tag{2.7}$$

ii) *There exists a constant C (independent of t) such that:*

$$\int_S K e^w \, dA \geq C. \tag{2.8}$$

iii) *For any small $\delta > 0$, there exists a constant $C_\delta > 0$ (independent of t) such that,*

$$0 < \mathsf{c} \leq C_\delta, \tag{2.9}$$

for any $t \geq \delta$.

Proof. Since the righthand side of (2.1) is uniformly bounded in L^1-norm because of (2.2), independent of t, then (2.7) follows from Stampacchia elliptic estimates. In particular, for any $p \geq 1$, there exists a constant $C_p > 0$ such that $\|w\|_p \leq C_p$. Denote by $\kappa_0 = \frac{\|K\|_\infty}{\|K\|_1}$, then by Jensen's inequality we find:

$$\begin{aligned} \int_S K(z) e^w dA &\geq \left(\int_S K \, dA \right) e^{\int_S \frac{K}{\|K\|_1} w \, dA} \\ &\geq \|K\|_1 e^{-\kappa_0 \|w\|_1} \\ &\geq e^{-C}, \end{aligned}$$

and (2.8) follows.

From (2.5), for $t \geq \delta$, we have:

$$\begin{aligned} 0 \leq \mathsf{c} &\leq \log\left(\frac{4\pi(g-1)}{t^2 \int_S K(z) e^w dA}\right) \\ &\leq \log\left(\frac{4\pi(g-1)}{\delta^2}\right) - \log\left(\int_S K(z) e^w dA\right), \end{aligned}$$

and (2.9) follows from (2.8). □

In the next section we complete the information of Lemma 2.4 by means of a more accurate blow-up analysis.

3. Blow-up Analysis

In this section we will study in details the general asymptotic behavior of a blow-up sequence of solution for problem $(1)_t$. As an application, we will prove Theorems B and C in sections §5 and §6.

Let us denote by $G(q,p)$ the Green's function (for the hyperbolic Laplace operator) defined as follows:

$$\begin{cases} -\Delta G = \delta_p - \frac{1}{4\pi(g-1)} \\ \int_S G(q,p)dA(q) = 0. \end{cases} \tag{3.1}$$

Here δ_p is the Dirac measure with pole at the point $p \in S$. It is well-known (see for instance [**Aub98**]) that

$$G(p,q) = G(q,p), \text{ for } p \neq q,$$

and

$$G(q,p) = -\frac{1}{2\pi}\log(dist(q,p)) + \gamma(q,p), \tag{3.2}$$

where $\gamma \in C^\infty(S \times S)$ is the regular part of the Green's function G.

3.1. **Mean field formulation.** Our blow-up analysis about solutions of problem $(1)_t$ is based on well-known results concerning blow-up solutions of Liouville type equations in mean field form (see [**BM91, LS94, BT02**]). Thus, we proceed first to reformulate problem $(1)_t$ to a mean field type equation. To this end, we let v be a solution of problem $(1)_t$ and set:

$$\rho = t^2 \int_S K e^v dA. \tag{3.3}$$

By the conservation identity (2.2), we know that $\rho \in (0, 4\pi(g-1))$. As before (in Lemma 2.2), we set $v = w + \mathsf{c}$, where $\int_S w(z)dA = 0$, and $\mathsf{c} = \fint_S v(z)dA$.

Lemma 3.1. *If $v = w + \mathsf{c}$ is a solution of the problem $(1)_t$ satisfying (3.3) with $\rho \in (0, 4\pi(g-1))$, then w satisfies:*

$$\begin{cases} -\Delta w = 2\rho\left(\frac{K(z)e^w}{\int_S K(z)e^w dA} - \frac{1}{|S|}\right) + 2(4\pi(g-1)-\rho)\left(\frac{e^{-w}}{\int_S e^{-w}dA} - \frac{1}{|S|}\right) \\ \int_S w(z)dA = 0, \end{cases} \tag{3.4}$$

Vice versa, if w_ρ solves equation (3.4), with $\rho \in (0, 4\pi(g-1))$, then by setting

$$\mathsf{c}_\rho = \log\left(\frac{\int_S e^{-w}dA}{4\pi(g-1)-\rho}\right),$$

and

$$t_\rho^2 = \frac{\rho(4\pi(g-1)-\rho)}{(\int_S K(z)e^w dA)(\int_S e^{-w}dA)},$$

we see that $v = w_\rho + \mathsf{c}_\rho$ is a solution to problem $(1)_{t_\rho}$.

Proof. This follows by direct and simple calculations. □

3.2. **General blow-up.** Recall that $K(z) = \frac{|\alpha(z)|^2}{det(g_\sigma)}$. Denote by $\{q_1, \cdots, q_N\}$ the (finite) set of distinct zeroes of α, i.e.,

$$\alpha(z) = 0 \iff z = q_i \text{ for some } i \in \{1, \cdots, N\},$$

and let n_i be the multipilcity of q_i. It is well known that, $\sum_{i=1}^{N} n_i = 4(g-1)$.

Our main result in this section is the following theorem, which in particular indicates that blow-up occurs only when $t \to 0$.

Theorem D. *Let v_n be a solution of the problem $(1)_{t_n}$ such that*

$$\max_S v_n \to +\infty, \text{ as } n \to \infty,$$

then, as $n \to \infty$,

$$t_n \to 0, \tag{3.5}$$

and

$$t_n^2 \int_S K(z) e^{v_n} \, dA \to 4\pi m, \quad \text{for some } m \in \{1, \cdots, g-1\}, \tag{3.6}$$

where $g \geq 2$ is the genus of S. Furthermore,

(i) *if $1 \leq m < g-1$, then (along a subsequence),*

(a) there exist $\{p_1, \cdots, p_s\} \subset S$ (called blow-up points), and sequences $\{p_{j,n}\} \subset S$ such that $p_{j,n} \to p_j$ and $v_n(p_{j,n}) \to +\infty$ as $n \to +\infty$, $j = 1, \cdots, s$. Moreover,

$$t_n^2 K(z) e^{v_n} \rightharpoonup 4\pi \sum_{j=1}^{s} (1 + n(p_j)) \delta_{p_j} \tag{3.7}$$

weakly in the sense of measure, with

$$n(p_j) = \begin{cases} 0, & \text{if } \alpha(p_j) \neq 0 \\ n_i, & \text{if } \alpha(p_j) = 0 \text{ for some } 1 \leq j \leq s, \text{ and } p_j = q_i, \end{cases} \tag{3.8}$$

and $m = \sum_{j=1}^{s} (1 + n(p_j))$.

(b) $v_n \rightharpoonup v_0$ weakly in $W^{1,q}(S)$, for some v_0 and $1 < q < 2$, and uniformly in $C^{2,\beta}_{loc}(S \backslash \{p_1, \cdots, p_s\}$ for some $\beta \in (0,1)$. We also have

$$e^{-v_n} \to e^{-v_0}, \tag{3.9}$$

strongly in $L^p(S)$ for all $p \geq 1$, and v_0 is the unique solution of the following equation on S:

$$-\Delta v_0 = 2\left(4\pi \sum_{j=1}^{s} (1 + n(p_j)) \delta_{p_j} + e^{-v_0} - 1\right). \tag{3.10}$$

(ii) *If $m = g - 1$, then by setting $v_n = w_n + \mathsf{c}_n$, where $\int_S w_n(z)dA = 0$, and $\mathsf{c}_n = \fint_S v_n(z)dA$, we have (along a subsequence):*

$$\mathsf{c}_n \to +\infty, \text{ and } w_n \rightharpoonup w_0 \text{ weakly in } W^{1,q}(S), 1 < q < 2. \tag{3.11}$$

Furthermore we have the following alternatives:

(a) *(Compactness) <u>either</u>,*

$$w_n \to w_0, \quad \text{strongly in } \; H^1(S), \text{ and in any other relevant norm,}$$

and

$$t_n^2 K(z)e^{v_n} \to \frac{4\pi(g-1)K(z)e^{w_0}}{\int_S K(z)e^{w_0}\, dA} \qquad \text{strongly in } H^1(S),$$

with w_0 satisfying the following equation on S:

$$\begin{cases} -\Delta w_0 = 8\pi(g-1)\left(\frac{K(z)e^{w_0}}{\int_S K(z)e^{w_0}dA} - \frac{1}{|S|}\right) \\ \int_S w_0(z)dA = 0, \end{cases}$$

(b) *(Concentration) <u>or</u>, for suitable points $\{p_1, \cdots, p_s\} \subset S$ (blow-up points), we have sequences $\{p_{i,n}\} \subset S$: $p_{i,n} \to p_i$ such that:*

$$w_n(p_{i,n}) = v_n(p_{i,n}) - \fint_S v_n \to +\infty, \quad \text{as } n \to +\infty,$$

and:

$$t_n^2 K(z)e^{v_n} \rightharpoonup 4\pi \sum_{i=1}^{s} (1 + n(p_i))\delta_{p_i},$$

weakly in the sense of measure, where $n(p_i)$ is defined in (3.8) with $\sum_{i=1}^{s}(1 + n(p_i)) = g - 1$, and $w_0(z)$ satisfying:

$$w_0(z) = 8\pi \sum_{i=1}^{s} (1 + n(p_i))G(z, p), \tag{3.12}$$

with $G(z, p)$ the unique Green's function in (3.1). The convergence is uniform in $C^{2,\beta}_{loc}(S \backslash \{p_1, \cdots, p_s\})$.

Proof. As in the statement, we write $v_n = w_n + \mathsf{c}_n$, where $\int_S w_n(z)dA = 0$, and $\mathsf{c}_n = \fint_S v_n(z)dA$. By Lemma 2.4, we can always assume that, along a subsequence,

$$w_n \rightharpoonup w_0 \text{ weakly in } W^{1,q}(S), \; 1 < q < 2.$$

Let

$$\rho_n = t_n^2 \int_S K e^{v_n} dA \in (0, 4\pi(g-1)). \tag{3.13}$$

So we can write $\int_S e^{-v_n} dA = 4\pi(g-1) - \rho_n$ (recall (2.2)).

Denote by ζ_n the unique solution to the problem:

$$\begin{cases} -\Delta \zeta_n = 2(4\pi(g-1) - \rho_n)\left(\frac{e^{-w_n}}{\int_S e^{-w_n}} - \frac{1}{|S|}\right) \quad \text{on } S, \\ \int_S \zeta_n(z)dA = 0. \end{cases} \tag{3.14}$$

Recall that $v_n > 0$ in S, and

$$(4\pi(g-1) - \rho_n)\frac{e^{-w_n}}{\int_S e^{-w_n}} = e^{-v_n}.$$

Therefore, we know that the righthand side of (3.14) is uniformly bounded in $L^\infty(S)$. Thus, by elliptic estimates, we derive that ζ_n is uniformly bounded in $C^{2,\beta}(S)$-norm, with $\beta \in (0,1)$. So, along a subsequence, we can assume that,

$$\zeta_n \to \zeta_0 \quad \text{in } C^2(S)\text{-norm}, \quad \text{as } n \to +\infty. \tag{3.15}$$

We define, somewhat abusing our notation to use z_n as follows,

$$z_n = w_n - \zeta_n, \tag{3.16}$$

which satisfies the following mean field type equation:

$$\begin{cases} -\Delta z_n = 2\rho_n \left(\frac{K_n(z)e^{z_n}}{\int_S K_n(z)e^{z_n}} - \frac{1}{|S|} \right) & \text{on } S, \\ \int_S z_n(z) dA = 0, \end{cases} \tag{3.17}$$

with

$$K_n = Ke^{\zeta_n} \to Ke^{\zeta_0} \quad \text{in } C^2(S), \quad \text{as } n \to +\infty. \tag{3.18}$$

At this point, we recall the following well-known "concentration-compactness" result of [**BM91, LS94, BT02**] which we state in a form suitable for our situation:

Theorem 3.2. *([**BM91, LS94, BT02**]) Assume* (3.17) *with* $\rho_n \in (0, 4\pi(g-1))$ *and* (3.18) *with* $K(z) = \frac{|\alpha|^2}{det(g_\sigma)}$*, then, along a subsequence, as* $n \to +\infty$*:*

$$z_n \rightharpoonup z_0 \text{ weakly in } W^{1,q}(S), 1 < q < 2,$$

and the following alternative holds:

(1) *<u>either</u>,* $\max_S z_n \le C$*, and along a subsequence, as* $n \to +\infty$*:*

$$\rho_n \to \rho_0 \in [0, 4\pi(g-1)],$$

$$z_n \to z_0 \quad \text{strongly in } H^1(S),$$

and in any other relevant norm, with z_0 *satisfying:*

$$\begin{cases} -\Delta z_0 = 2\rho_0 \left(\frac{he^{z_0}}{\int_S he^{z_0}} - \frac{1}{|S|} \right) & \text{on } S, \\ \int_S z_0 dA = 0, \end{cases} \tag{3.19}$$

with $h = Ke^{\zeta_0}$ *(see* (3.15)*).*

(2) *<u>or</u>, there exist (blow-up) points* $\{p_1, \cdots, p_s\} \subset S$*, and sequences* $\{p_{j,n}\} \subset S$*:* $p_{j,n} \to p_j$*, such that* $z_n(p_{j,n}) \to +\infty$ *as* $n \to +\infty$*, such that,*

$$\rho_n \frac{K_n(z)e^{z_n}}{\int_S K_n(z)e^{z_n}} \rightharpoonup 4\pi \sum_{j=1}^{s} (1 + n(p_j))\delta_{p_j}, \tag{3.20}$$

weakly in the sense of measure, with $n(p_j)$ *defined in* (3.8)*. In particular,*

$$\rho_n \to \rho_0 = 4\pi \sum_{j=1}^{s} (1 + n(p_j)) \in 4\pi\{1, \cdots, (g-1)\} \subset 4\pi\mathbb{N}, \tag{3.21}$$

$$z_n \to z_0 \quad \textit{uniformly in } C^{2,\beta}_{loc}(S\backslash\{p_1,\cdots,p_s\}), \quad 0<\beta<1,$$

and

$$z_0(x) = 8\pi \sum_{i=1}^{s}(1+n(p_i))G(x,p_i).$$

Proof. See Theorem 5.7.65 in [**Tar08**]. □

Back to the proof of Theorem D. We apply Theorem 3.2 to z_n in (3.16), and along a subsequence, we have:

$$z_n \rightharpoonup z_0 \text{ weakly in } W^{1,q}(S),\ 1<q<2.$$

Recall that by assumption: $\max_S v_n \to \infty$, and so in case alternative (1) in Theorem 3.2 holds, in view of (3.15) and (3.16), we see that necessarily

$$\mathsf{c}_n \to +\infty, \text{ and } w_n \to w_0 \text{ strongly }, \quad \text{as } n\to\infty.$$

Therefore, by dominated convergence, we derive:

$$4\pi(g-1)-\rho_n = \int_S e^{-v_n}dA \to 0, \text{ as } n\to\infty. \tag{3.22}$$

So $\rho_n \to 4\pi(g-1)$, and by (3.13) we deduce that (3.6) must hold with $m=g-1$ in this case.

This covers the compactness part in the statement (ii). Furthermore, by part (iii) of Lemma 2.4, we see also that $t_n\to 0$, as $n\to\infty$, and so (3.5) holds.

Next we assume that alternative (2) in Theorem 3.2 holds. Then by virtue of (3.13), (3.20), and (3.21), we check that (3.6) holds with $m := \sum_{j=1}^{s}(1+n(p_j))$. We consider first the case where $1\le m< g-1$. As a consequence,

$$\int_S e^{-v_n} \to 4\pi(g-1-m)>0,$$

and so necessarily $\mathsf{c}_n = \fint_S v_n$ must be uniformly bounded. Thus along a subsequence, we have: $v_n \rightharpoonup v_0$ weakly in $W^{1,q}(S)$, $1<q<2$, and also uniformly in $C^{2,\beta}(S\backslash\{p_1,\cdots,p_s\})$ with $0<\beta<1$, for some v_0. Furthermore, by dominated convergence, we see that

$$e^{-v_n}\to e^{-v_0} \quad \text{in } L^p(S),\ \forall\, p\ge 1.$$

As a consequence v_0 satisfies (3.10). In other words, we have verified that part (i) holds in this case.

Since,

$$v_n\to v_0, \text{ and } t_n^2Ke^{v_n}\to 0, \text{ as } n\to\infty,$$

uniformly on compact sets of $S\backslash\{p_1,\cdots,p_s\}$, we may conclude that (3.5) must hold as well.

Finally, when we have alternative (2) with $m=g-1$, then necessarily

$$\int_S e^{-v_n}\to 0, \text{ as } n\to\infty.$$

As a consequence, $\mathsf{c}_n\to+\infty$, as $n\to\infty$ and this implies as above, that $t_n\to 0$, by part (c) of Lemma 2.4. Thus we have verified (3.5), (3.11), and (3.6) with

$m = g-1$. At this point, alternative (2) of Theorem 3.2 in this situation gives exactly the (concentration) part (b) of (ii).

Finally, since w_0 in (3.11) satisfies:

$$\begin{cases} -\Delta w_0 = 8\pi \sum_{j=1}^{s}(1+n(p_j))\delta_{p_j} - 2 = \sum_{j=1}^{s} 8\pi(1+n(p_j))(\delta_{p_j} - \frac{1}{|S|}) \\ \int_S w_0 = 0, \end{cases} \tag{3.23}$$

we see that (3.12) holds, and the proof is complete. □

Concerning the location of the (possible) blow-up points $\{p_1, \cdots, p_s\}$ of v_n, we can use well-known results ([**OS05**]) which apply to the blow-up points of the sequence z_n in (3.17). Thus, according to Theorem 2.2 of ([**OS05**]), we conclude that, if p_j is a blow-up point with $\alpha(p_j) \neq 0$, then in conformal coordinates around p_j there holds:

$$\nabla_z \left(8\pi\gamma(z,p_i) + 8\pi \sum_{j\neq i}(1+n(p_i))G(z,p_j) + \log h \right) |_{z=p_i} = 0, \tag{3.24}$$

with $i \in \{1, \cdots, s\}$, and

$$h = Ke^{\zeta_0}, \tag{3.25}$$

with ζ_0 in (3.15) and $K = \|\alpha\|_\sigma^2$.

Notice in particular that when $m = g-1$, then the function $\zeta_0 \equiv 0$, and (3.24) provides a well-known necessary condition for blow-up at $\{p_1, \cdots, p_s\}$. Indeed, in case of non-degeneracy, (3.24) turns out to be also a sufficient condition for the construction of blow-up solutions at $\{p_1, \cdots, p_s\}$ for mean field equations on surfaces, see for instance [**CL03**].

On the contrary, when $1 \leq m < g-1$, the condition (3.24) is more involved since the function ζ_0 is nonzero and satisfies the following equation:

$$\begin{cases} -\Delta\zeta_0 = 8\pi(g-1-m)\left(\frac{e^{-8\pi \sum_{j=1}^{s}(1+n(p_j))G(z,p_j)} e^{-\zeta_0}}{\int_S e^{-8\pi \sum_{j=1}^{s}(1+n(p_j))G(z,p_j)} e^{-\zeta_0}\, dA} - \frac{1}{|S|} \right) \quad \text{on } S, \\ \int_S \zeta_0(z)\, dA = 0. \end{cases} \tag{3.26}$$

So ζ_0 itself depends on the blow-up points $\{p_1, \cdots, p_s\}$. Therefore it would be interesting to see whether one can find a (nondegenerate) set of points satisfying (3.24), (3.25) and (3.26) which turns out to be the blow-up set of a sequence of bubbling solutions for $(1)_t$, along a sequence of t's going to zero.

As a consequence of Theorem D, we know that blow-up can only occur as $t \to 0$. Therefore, we can complete the (uniform) estimates given in Lemma 2.4 as follows:

Corollary 3.3. *For any $\delta > 0$, there exists a constant $C_\delta > 0$ such that any solution v of the problem $(1)_t$ with $t \geq \delta$ satisfies:*

$$\|v\|_\infty \leq C_\delta.$$

Actually, by means of elliptic estimates we know that the $L^\infty(S)-$norm above can be replaced by any other stronger norm. Theorem D can be better interpreted in terms of the mean field formulation of problem $(1)_t$, and gives the following "compactness" result:

Corollary 3.4. *Let w_n be a sequence of solutions for* (3.4) *with $\rho = \rho_n$, and $\rho_n \to \rho_0 \in (0, 4\pi(g-1)]\backslash\{4\pi m, 1 \le m \le g-1\}$. Then along a subsequence, $w_n \to w_0$ in $H^1(S)$ (and any other relevant norm), with w_0 a solution of* (3.4) *with $\rho = \rho_0$.*

Corollary 3.5. *For every compact set $\mathcal{A} \subset (0, 4\pi(g-1)]\backslash\{4\pi m, 1 \le m \le g-1\}$, the set of solutions of* (3.4) *with $\rho \in \mathcal{A}$ is uniformly bounded in $C^{2,\beta}(S)$, $0 < \beta < 1$.*

4. Proof of theorem A

In this section, we will prove parts of Theorem A, in various steps. In this way, we obtain a detailed description of the lower branch of the bifurcation solution curve $\mathcal{C}$. We shall take advantage of the variational formulation of problem $(1)_t$. Indeed, it is easy to verify that (weak) solutions of problem $(1)_t$ correspond to the critical points of the following functional:

$$\mathcal{I}_t(v) = \frac{1}{2}\|\nabla v\|_2^2 - 2t^2\int_S K(z)e^v dA + 2\int_S e^{-v}dA + 2\int_S v dA,\ \forall\ v \in H^1(S). \tag{4.1}$$

4.1. **First bending point.** We define the following two sets:

$$\Lambda = \{t \ge 0 : (1)_t \text{ admits a solution }\}, \tag{4.2}$$

and

$$\Lambda_s = \{t \ge 0 : (1)_t \text{ admits a } \textit{stable} \text{ solution }\}. \tag{4.3}$$

Clearly $\Lambda_s \subseteq \Lambda \subset [0, t^*]$, with t^* given in (2.6). Furthermore, since the problem $(1)_{t=0}$ only admits the trivial solution $v = 0$, which is strictly stable, we see that, Λ_s is nonempty and,

$$0 < \tau_0 = \sup\{\Lambda_s\} \le t_0 = \sup\{\Lambda\}. \tag{4.4}$$

We aim to show that $\Lambda = \Lambda_s$ and $t_0 = \tau_0$.

To this purpose, we observe firstly that, by the estimates in Corollary 3.3 and a limiting argument, we know that problem $(1)_{\tau_0}$ admits a stable solution v_0 which is also degenerate. According to the language of Crandell and Rabinowitz ([**CR80**]), (v_0, τ_0) defines a "bending point" for the curve of solutions of problem $(1)_t$, given by the zero set of the map

$$F(v,t) = \Delta v + 2 - 2(e^{-v} + t^2 K e^v) : C^{2,\beta}(S) \times \mathbb{R} \to C^{0,\beta}(S),$$

with $0 < \beta < 1$.

To establish Theorem 0.1, Uhlenbeck ([**Uhl83**]) showed that actually τ_0 is the only value for which the problem $(1)_t$ admits a degenerate stable solution.

Proposition 4.1. *The problem $(1)_t$ admits a degenerate stable solution only at $t = \tau_0$. Moreover, for any $t \in [0, \tau_0]$ problem $(1)_t$ admits a unique stable solution which forms a smooth monotone increasing curve (with respect to t), and it is strictly stable for $t \in [0, \tau_0)$.*

Proof. Let v_0 be the degenerate stable solution for $(1)_{\tau_0}$. We know that (v_0, τ_0) must correspond to a "bending point" for the set: $F(v, t) = 0$, around (v_0, τ_0) (see [**CR80**]).In other words, for $\epsilon > 0$ small, there exists a smooth curve $(v(s), t(s))$ satisfying $F(v(s), t(s)) = 0$ for all $s \in (-\epsilon, \epsilon)$, such that: $v(0) = v_0,\ t(0) = \tau_0$, $\dot{t}(0) = 0$, and $\dot{v} > 0$ (i.e. $v(s)$ is increasing), where we have used the "dot" to denote derivatives with respect to s.

Uhlenbeck showed further that $\ddot{t}(0) < 0$ (see [**Uhl83**]), so that:

$$t(s) < t_0,\ \forall\ s \in (-\epsilon, \epsilon)\backslash\{0\},$$

and in particular, $\dot{t}(s) > 0$ for $s \in (-\epsilon, 0)$. This implies that $v(s)$ is strictly stable for every $s \in (-\epsilon, 0)$.

Note that the same local description would hold for any other (possible) degenerate stable solution for which the corresponding of (4.3) would hold. This shows that if we continue the lower branch $(v(s), t(s)),\ s \in (-\epsilon, 0)$ with the Implicit Function Theorem, we see that it cannot join with another degenerate stable solution at lower value of t. Instead, the lower branch can be continued until it joins the trivial solution at $t = 0$. Thus, we can conclude that there exists a smooth, increasing curve of strictly stable solutions of problem $(1)_t,\ t \in [0, \tau_0)$, which joins the trivial solution $v = 0$ at $t = 0$ with the degenerate stable solution v_0 at $t = \tau_0$. This also shows that for any $t \in [0, \tau_0]$, problem $(1)_t$ cannot admit any other stable solution. As otherwise we could argue as before to join such a different solution to the trivial solution along another smooth curve of solutions, a contradiction to the non-denegeracy of the trivial solution $v \equiv 0$ at $t = 0$. This concludes the proof. □

Proposition 4.1 shows in particular that $\Lambda_s = [0, \tau_0]$ and it also furnishes a proof to Theorem 0.1.

In the next subsection, we shall prove, with the help of the sub/super solution method in variational guise (see [**Str00**]), that problem $(1)_t$ admits a stable solution for any $t \in [0, t_0]$ where $t_0 = \sup\{\Lambda\}$. Consequently $\Lambda = \Lambda_s = [0.\tau_0]$.

4.2. **Stable solutions.** We now prove the following theorem on stable solutions:

Theorem 4.2. *Let $t_0 = \sup \Lambda$ (Λ in (4.2)). For any $t \in [0, t_0]$, problem $(1)_t$ admits a* **stable solution** *$v_{1,t}$ which is strictly increasing with respect to t, and coincides with the smallest of the solutions (and supersolutions) of Problem $(1)_t$. Furthermore, for $t = t_0$,*

$$v_{1,t_0}(z) = \sup_{0 \leq t < t_0} v_{1,t}(z) < +\infty$$

is the <u>unique</u> solution for problem $(1)_{t_0}$. In particular: $t_0 = \tau_0$ and $\Lambda_s = \Lambda = [0, \tau_0]$.

Proof. Since for $t = 0$, $v \equiv 0$ is the desired stable solution, we let $t \in (0, t_0)$ be fixed. By (4.4), we can find some $t_1 \in \Lambda$ such that, $0 < t < t_1$. We denote by $v_1 > 0$

a solution for problem $(1)_{t_1}$, and observe that it defines a strict super-solution for problem $(1)_t$. Indeed, we have

$$\int_S \nabla v_1 \nabla\phi dA - 2t^2\int_S K(z)e^{v_1}\phi dA + 2\int_S e^{-v_1}\phi dA + 2\int_S v_1\phi dA > 0,$$

for any $\phi \in H^1(S)$ with $\phi \geq 0$ a.e. in S, and $\phi \not\equiv 0$.

While $v_0 \equiv 0$ is an obvious strict sub-solution for problem $(1)_t$. We set

$$\mathcal{Z} = \{v \in H^1(S) : 0 \leq v \leq v_1 \text{ a.e. in } S\}. \tag{4.5}$$

It is routine to verify that $\mathcal{Z}$ is a non-empty, convex and closed subset of $H^1(S)$. In addition, the functional $\mathcal{I}_t$ is bounded from below and coercive on $\mathcal{Z}$. Consequently the functional $\mathcal{I}_t$ attains its minimum value at a point $v_{1,t}$ in $\mathcal{Z}$, i.e.,

$$\mathcal{I}_t(v_{1,t}) = \min_{\mathcal{Z}} \mathcal{I}_t, \quad \text{and } 0 \leq v_{1,t} \leq v_1 \text{ on } S. \tag{4.6}$$

By the strict sub/super solution property of $v_0 \equiv 0$ and v_1 respectively, we have

$$\mathcal{I}_t(v_{1,t}) < \min\{\mathcal{I}_t(0), \mathcal{I}_t(v_1)\},$$

and therefore $v_{1,t} \not\equiv v_1$ and $v_t \not\equiv 0$. Following [**Str00**], we show next that $v_{1,t}$ is a critical point for $\mathcal{I}_t$, therefore a solution to $(1)_t$. We first let $\phi \in C^\infty(S)$ with $\phi \geq 0$, and for $\epsilon > 0$ sufficiently small, we define

$$v_\epsilon = v_{1,t} + \epsilon\phi - \phi^\epsilon + \phi_\epsilon, \tag{4.7}$$

where

$$\phi^\epsilon = \max\{0, v_{1,t} + \epsilon\phi - v_1\} \geq 0,$$

and

$$\phi_\epsilon = \max\{0, -(v_{1,t} + \epsilon\phi)\} \geq 0.$$

Therefore we have $v_\epsilon \in \mathcal{Z}$. By virtue of (4.6) and (4.7), we have:

$$\begin{aligned} 0 &\leq \langle \mathcal{I}_t'(v_{1,t}), v_\epsilon - v_{1,t}\rangle \\ &= \epsilon\langle \mathcal{I}_t'(v_{1,t}), \phi\rangle - \langle \mathcal{I}_t'(v_{1,t}), \phi^\epsilon\rangle + \langle \mathcal{I}_t'(v_{1,t}), \phi_\epsilon\rangle. \end{aligned} \tag{4.8}$$

We define a set

$$\Omega_\epsilon = \{p \in S : v_{1,t}(p) + \epsilon\phi(p) \geq v_1(p) > v_{1,t}(p)\}.$$

We observe that, $|\Omega_\epsilon|$ the measure of Ω_ϵ goes to zero as $\epsilon \to 0$. Moreover,

$$\begin{aligned} -\langle \mathcal{I}_t'(v_{1,t}), \phi^\epsilon\rangle &= -\langle \mathcal{I}_t'(v_1), \phi^\epsilon\rangle - \langle \mathcal{I}_t'(v_{1,t}) - \mathcal{I}_t'(v_1), \phi^\epsilon\rangle \\ &\leq -\int_S \nabla(v_{1,t} - v_1)\nabla\phi^\epsilon + 2t^2\int_S K(e^{v_{1,t}} - e^{v_1})\phi^\epsilon \\ &\qquad -2\int_S (e^{-v_{1,t}} - e^{-v_1})\phi^\epsilon - 2\int_S (v_{1,t} - v_1)\phi^\epsilon \\ &\leq -\int_S \nabla(v_{1,t} - v_1)\nabla\phi^\epsilon - 2\int_S (v_{1,t} - v_1)\phi^\epsilon \\ &= -\int_{\Omega_\epsilon} \nabla(v_{1,t} - v_1)\nabla(v_{1,t} - v_1 + \epsilon\phi) - 2\int_{\Omega_\epsilon} (v_{1,t} - v_1)\phi^\epsilon \\ &\leq -\int_{\Omega_\epsilon} \nabla(v_{1,t} - v_1)\nabla(v_{1,t} - v_1 + \epsilon\phi) \end{aligned}$$

$$
\begin{aligned}
& +2\int_{\Omega_\epsilon}(e^{-v_{1,t}}-e^{-v_1})\phi^\epsilon+2\int_{\Omega_\epsilon}(e^{-v_{1,t}}-e^{-v_1})\phi^\epsilon \\
\leq\ & -\epsilon\int_{\Omega_\epsilon}\nabla(v_{1,t}-v_1)\nabla\phi+2\epsilon\int_{\Omega_\epsilon}(v_1-v_{1,t})\phi \\
=\ & o(\epsilon) \quad \text{as } \epsilon\to 0.
\end{aligned}
$$

Similar calculations show that

$$\langle \mathcal{I}_t'(v_{1,t}),\phi_\epsilon\rangle = o(\epsilon), \quad \text{as } \epsilon\to 0.$$

Applying (4.8), we find:

$$\langle \mathcal{I}_t'(v_{1,t}),\phi\rangle \geq 0.$$

We obtain the reverse inequality by replacing ϕ with $-\phi$. Since $C^\infty(S)$ is dense in $H^1(S)$, by a density argument we have now proved:

$$\langle \mathcal{I}_t'(v_{1,t}),\phi\rangle = 0, \forall\phi\in H^1(S).$$

This implies that $v_{1,t}$ is a solution to problem $(1)_t$. Note that $v_{1,t}\not\equiv 0$ and $v_{1,t}\not\equiv v_1$, so by the maximum principle, we have:

$$0<v_{1,t}(z)<v_1(z), \forall z\in S. \tag{4.9}$$

This ensures that $v_{1,t}$ is a local minimum for the functional $\mathcal{I}_t$ in $C^1(S)$-norm.

We are going to show that $v_{1,t}$ is actually a local minimum for $\mathcal{I}_t$ in $H^1(S)$-norm, and hence a stable solution for problem $(1)_t$. To this purpose, we argue by contradiction. Suppose there exists $v_n\in H^1(S)$ such that $\mathcal{I}_t(v_n)<\mathcal{I}_t(v_{1,t})$, and $v_n\to v_{1,t}$ in $H^1(S)$. Letting

$$\delta_n^2=\frac{1}{2}\|\nabla(v_{1,t}-v_n)\|_2^2+\|v_{1,t}-v_n\|_2^2\to 0, \quad \text{as } n\to+\infty,$$

we may assume, without loss of generality, that

$$\mathcal{I}_t(v_n)=\min_{v\in H^1(S)}\{\mathcal{I}_t(v):\frac{1}{2}\|\nabla(v-v_{1,t})\|_2^2+\|v-v_{1,t}\|_2^2=\delta_n^2\}.$$

This enables us to apply the Lagrange multiplier method, and for suitable $\lambda_n\in\mathbb{R}$, we find that v_n satisfies

$$-\Delta v_n = 2t^2K(z)e^{v_n}+2e^{-v_n}-2+\lambda_n(-\Delta(v_n-v_{1,t})+2(v_n-v_{1,t})) \tag{4.10}$$

We set

$$\eta_n=\frac{v_n-v_{1,t}}{\sqrt{\|\nabla(v_{1,t}-v_n)\|_2^2+2\|v_{1,t}-v_n\|_2^2}}. \tag{4.11}$$

Then we see that $\|\eta_n\|_{H^1}\leq 1$. So along a subsequence, η_n converges weakly in $H^1(S)$ to some function $\eta\in H^1(S)$, as $n\to+\infty$. Moreover, for any $p\geq 1$ we also have

$$\|\eta_n-\eta\|_p\to 0. \tag{4.12}$$

By recalling that $\mathcal{I}_t'(v_{1,t})=0$, we compute:

$$
\begin{aligned}
0 &> \mathcal{I}_t(v_n)-\mathcal{I}_t(v_{1,t}) \\
&= \mathcal{I}_t(v_n)-\mathcal{I}_t(v_{1,t})+\mathcal{I}_t'(v_{1,t})(v_{1,t}-v_n)
\end{aligned}
$$

$$\begin{aligned} &= 2t^2 \int_S K e^{v_{1,t}}(1 + v_n - v_{1,t} - e^{v_n - v_{1,t}})dA \\ &\quad +2\int_S e^{-v_{1,t}}(e^{v_{1,t}-v_n} - 1 - (v_{1,t} - v_n))dA + \frac{1}{2}\|\nabla(v_n - v_{1,t})\|_2^2. \end{aligned} \tag{4.13}$$

Since $v_n \to v_{1,t}$ in $H^1(S)$, we know that, for any $p \geq 1$,

$$e^{v_n - v_{1,t}} \to 1, \text{ in } L^p(S).$$

Therefore by view of (4.13) we may conclude that

$$\int_S K(z)e^{v_{1,t}} \frac{(e^{v_n - v_{1,t}} - 1 - (v_n - v_{1,t}))}{\|\nabla(v_{1,t} - v_n)\|_2^2 + 2\|v_{1,t} - v_n\|_2^2} \to \frac{1}{2}\int_S K(z)e^{v_{1,t}}\eta^2, \tag{4.14}$$

and

$$\int_S e^{-v_{1,t}} \frac{(e^{v_n - v_{1,t}} - 1 - (v_n - v_{1,t}))}{\|\nabla(v_{1,t} - v_n)\|_2^2 + 2\|v_{1,t} - v_n\|_2^2} \to \frac{1}{2}\int_S e^{-v_{1,t}}\eta^2. \tag{4.15}$$

We claim that $\eta \neq 0$. To see this, by contradiction we assume $\eta = 0$, then

$$\|v_n - v_{1,t}\|_2 = o(\|\nabla(v_n - v_{1,t})\|_2),$$

as $n \to \infty$. From (4.13), (4.14) and (4.15), we find:

$$\begin{aligned} 0 &\geq \lim_{n\to\infty} \frac{\mathcal{I}_t(v_n) - \mathcal{I}_t(v_{1,t})}{\|\nabla(v_{1,t} - v_n)\|_2^2 + 2\|v_{1,t} - v_n\|_2^2} \\ &= \frac{1}{2} - t^2\int_S K(z)e^{v_{1,t}}\eta^2 + \int_S e^{-v_{1,t}}\eta^2 \\ &= \frac{1}{2}, \end{aligned} \tag{4.16}$$

which is impossible. Therefore $\eta \neq 0$.

Using (2.1) for $v_{1,t}$ and (4.10), we see that:

$$\begin{aligned} (1-\lambda_n)(-\Delta(v_n - v_{1,t}) + 2(v_n - v_{1,t})) &= 2t^2 K(z)(e^{v_n} - e^{v_{1,t}}) \\ &\quad +2(e^{-v_n} - e^{-v_{1,t}}) + 2(v_n - v_{1,t}). \end{aligned} \tag{4.17}$$

As above we find, as $n \to \infty$,

$$\int_S K(z)(e^{v_n} - e^{v_{1,t}})\frac{(v_n - v_{1,t})}{\|\nabla(v_{1,t} - v_n)\|_2^2 + 2\|v_{1,t} - v_n\|_2^2} \to \int_S K(z)e^{v_{1,t}}\eta^2, \tag{4.18}$$

and

$$\int_S (e^{-v_n} - e^{-v_{1,t}})\frac{(v_n - v_{1,t})}{\|\nabla(v_{1,t} - v_n)\|_2^2 + 2\|v_{1,t} - v_n\|_2^2} \to -\int_S e^{-v_{1,t}}\eta^2. \tag{4.19}$$

Now we have:

$$\lim_{n\to\infty}(1-\lambda_n) = 2t^2\int_S K(z)e^{v_{1,t}}\eta^2 + 2\int_S (1 - e^{-v_{1,t}})\eta^2 > 0. \tag{4.20}$$

So by (4.17) and (4.20), we can use elliptic regularity theory to conclude that $v_n \in C^1(S)$. Furthermore, the righthand side of (4.17) converges to zero in $L^p(S)$, for $p > 1$, as $n \to +\infty$. Consequently, by (4.20), we can use again elliptic estimates to show that $(v_n - v_{1,t}) \to 0$ in C^1-norm. This is a contradiction to the fact that $v_{1,t}$ is a local minimizer for the functional $\mathcal{I}_t$ in C^1-norm. Therefore $v_{1,t} \in H^1(S)$

is a local minimum of $\mathcal{I}_t$ and hence a *stable* solution for the problem $(1)_t$ with $t \in (0, t_0)$.

So far we have shown that $\forall t \in [0, t_0)$, problem $(1)_t$ admits a stable solution $v_{1,t}$ (and infinitely many supersolutions). As a consequence,

$$t_0 = \tau_0 = \sup \Lambda_s,$$

and by Proposition 4.1 we know also that problem $(1)_{\tau_0}$ admits a unique stable (degenerate) solution v_0.

For $t \in (0, \tau_0)$, to show that $v_{1,t}$ is the smallest among all solutions (and supersolutions) of problem $(1)_t$, we define for $z \in S$,

$$v_t(z) = \inf\{v(z) : v \text{ a solution or supersolution of problem } (1)_t\} \geq 0. \tag{4.21}$$

Clearly, $v_t(z)$ defines a supersolution of problem $(1)_t$, in the sense that the following holds:

$$\int_S \nabla v \nabla \phi - 2t^2 \int_S K(z) e^v \phi - 2 \int_S e^{-v} \phi + 2 \int_S \phi \geq 0, \tag{4.22}$$

for any $\phi \in H^1(S)$ with $\phi \geq 0$. Since $t > 0$, we have $v_t \not\equiv 0$. Moreover, (4.22) can never hold with a strict sign, as otherwise we would be in position to apply the sub/super solution method as above, and obtain a solution of problem $(1)_t$ which is smaller than v_t, in contradiction with (4.21). Hence v_t is a solution of problem $(1)_t$ which, by definition, is the smallest solution of $(1)_t$ and strictly increasing with respect to $t \in [0, \tau_0)$.

We claim:

$$v_t = v_{1,t}. \tag{4.23}$$

To establish this claim, it suffices to show that v_t is stable, so that (4.23) follows by the uniqueness of stable solutions in Proposition 4.1.

To this purpose, for $t \in (0, \tau_0)$, we use v_t as a supersolution to problem $(1)_s$, for $0 < s < t$. As above, for $s \in (0, t)$, we obtain a stable solution $\bar{v}_s$ of problem $(1)_s$ satisfying $0 < \bar{v}_s < v_t$ in S. By taking a sequence $s_n \nearrow t$, then by dominated convergence and elliptic estimates, we see that $\bar{v}_{s_n} \to \bar{v}$ in $H^1(S)$, with $\bar{v}$ a stable solution of problem $(1)_t$ and $0 < \bar{v} \leq v_t$. Since v_t is the smallest solution to $(1)_t$, we conclude that $\bar{v} \equiv v_t$, and so v_t is stable and (4.23) is established.

From (4.23), it also follows that, $\forall t \in [0, \tau_0)$, $v_{1,t} < v_0$, with v_0 the unique stable solution of problem $(1)_{\tau_0}$. So by the monotonicity property of $v_{1,t}$ in t, we find:

$$\lim_{t \nearrow \tau_0} v_{1,t}(z) = \sup_{0 \leq t < \tau_0} \{v_{1,t}(z)\} = v_0(z), \text{ as } t \nearrow \tau_0,$$

where again by dominated convergence and elliptic estimates, the convergence actually occurs uniformly in $C^{2,\beta}(S), 0 < \beta < 1$. Clearly, v_0 must define the smallest solution of problem $(1)_{\tau_0}$, i.e., $v_0 = v_{1,\tau_0}$. In fact we show that actually v_0 is the only solution of problem $(1)_{\tau_0}$.

To this purpose we argue by contradiction and assume there is another solution v' for the problem $(1)_{\tau_0}$. By construction, v_0 is the smallest solution at $t = \tau_0$, so necessarily $v' > v_0$ on S. As seen in Proposition 4.1, around (v_0, τ_0), we find a solution curve $(v(s), t(s))$ such that for $s \in (0, \epsilon)$ and $\epsilon > 0$ sufficiently small, we obtain a solution $v(s)$ for the problem $(1)_{t(s)}$ such that $t(s) < \tau_0$ and $v_0 < v(s) < v'$. Thus for $t \in (t(s), \tau_0)$ we find $v(s)$ as subsolution and v' as supersolution for problem $(1)_t$. So we can use the sub/super solution method again, and for $t \in (t(s), \tau_0)$, we obtain a stable solution for the problem $(1)_t$ which will be greater than v_0, and therefore greater than the smallest solution $v_{1,t}$. This is impossible, since the smallest solution is also the only (strictly) stable solution of problem $(1)_t$. □

4.3. **Compactness for the functional $\mathcal{I}_t$.** We will complete the proof of Theorem A in this subsection, namely we will prove the existence of an additional solution for each $t \in (0, t_0)$. This will extend a multiplicity result in ([**HL12**]).

By combining Proposition 4.1 and Theorem 4.2, we have now established that, $\forall\ t \in (0, \tau_0)$, $v_{1,t}$ is a strict local minimum for $\mathcal{I}_t(v)$ in $H^1(S)$. Furthermore, one checks that, for every $t \in (0, \tau_0)$:

$$\mathcal{I}_t(v_{1,t} + C) \to -\infty,$$

as $C \to +\infty$. In other words, for $t \in (0, \tau_0)$, the functional $\mathcal{I}_t$ admits a “mountain-pass” structure ([**AR73**]). Next we establish the following Palais-Smale (compactness) condition:

Theorem 4.3. *Suppose a sequence $\{v_n\} \in H^1(S)$ satisfies that, $\mathcal{I}_t(v_n) \to c$ and $\mathcal{I}_t'(v_n) \to 0$ as $n \to \infty$, then $\{v_n\}$ admits a convergent subsequence. In particular, c is a critical value for the functional $\mathcal{I}_t$.*

Proof. We show first that v_n is uniformly bounded in H^1-norm. As in Lemma 2.2, we write $v_n = w_n + \mathsf{c}_n$, with $\int_S w_n(z)dA = 0$, and $\mathsf{c}_n = \fint_S v_n(z)dA$. Then, we have

$$\langle \mathcal{I}_t'(v_n), 1\rangle = -2t^2 e^{\mathsf{c}_n} \int_S K(z) e^{w_n} dA - 2e^{-\mathsf{c}_n} \int_S e^{-w_n} dA + 8\pi(g-1). \tag{4.24}$$

By assumption, $\langle \mathcal{I}_t'(v_n), 1\rangle = o(1)$ as $n \to \infty$. Applying Jensen's inequality, we have, as $n \to \infty$:

$$e^{-\mathsf{c}_n} \le e^{-\mathsf{c}_n} \fint_S e^{-w_n} dA \le 1 + o(1).$$

Therefore for some suitable constant $C_0 > 0$, we find $\mathsf{c}_n \ge -C_0$. Now from (4.1) and (4.24), we obtain, as $n \to \infty$:

$$\mathcal{I}(v_n) = \frac{1}{2}\|\nabla w_n\|_2^2 + 4e^{-\mathsf{c}_n} \int_S e^{-w_n} dA - 8\pi(g-1) + 8\pi(g-1)\mathsf{c}_n + o(1). \tag{4.25}$$

By assumption, $\mathcal{I}(v_n)$ is uniformly bounded, so from (4.25) we also see that c_n is bounded from above, and that $\|\nabla w_n\|_2$ is uniformly bounded.

In conclusion we have $\|v_n\|_{H^1} \le C$ for some suitable $C > 0$. Therefore along a subsequence, v_n converges to some $v \in H^1(S)$ weakly. The convergence is strong in $L^p(S)$ for $p \ge 1$. In particular we have $\mathsf{c}_n \to \fint_S v dA$ and $\mathcal{I}'(v) = 0$. By the Moser-Trudinger inequality, we also have:

$$\|e^{|v_n|}\|_{L^p} \le C_p, \forall p \ge 1.$$

In particular, for $p = 2$, and as $n \to \infty$,

$$o(1) \quad = \quad \langle \mathcal{I}'(v_n), v_n - v\rangle$$

$$\begin{aligned}
&= \langle \mathcal{I}'(v_n) - \mathcal{I}'(v), v_n - v \rangle \\
&= \|\nabla(v_n - v)\|_2^2 - 2t^2 \int K(e^{v_n} - e^v)(v_n - v) - 2\int_S (e^{-v_n} - e^{-v})(v_n - v) \\
&\geq \|\nabla(v_n - v)\|_2^2 - C\|v_n - v\|_2^2 \\
&= \|\nabla(v_n - v)\|_2^2 + o(1).
\end{aligned}$$

In other words, we have

$$\|\nabla(v_n - v)\|_2 \to 0,$$

as $n \to \infty$. This completes the proof. □

We can now apply the mountain pass construction of Ambrosetti-Rabinowitz ([**AR73**]) to obtain a second (unstable) mountain pass solution $v_{2,t} > v_{1,t}$ for all $t \in (0, t_0)$, satisfying:

$$\mathcal{I}_t(v_{2,t}) = \inf_{\Gamma \in \mathcal{P}_t} \max_{s \in [0,1]} \mathcal{I}_t(\Gamma(s)) > \mathcal{I}_t(v_{1,t}), \tag{4.26}$$

with the path space

(4.27)

$$\mathcal{P}_t = \{\Gamma : [0,1] \to H^1(S) \text{ is continuous with } \Gamma(0) = v_{1,t},\ \mathcal{I}_t(\Gamma(1)) \leq \mathcal{I}_t(v_{1,t}) - 10\}.$$

Clearly $\mathcal{P}_t$ is not empty, since for $A > 0$ sufficiently large we easily check that $\Gamma(s) = v_{1,t} + sA,\ s \in [0,1]$ lies in $\mathcal{P}_t$.

Finally we show that the unstable solution $v_{2,t}$ will not stay bounded as $t \to 0$:

Proposition 4.4. *For $t \in (0, \tau_0)$, let $v_{2,t}$ be the mountain pass solution obtained above. Then:*

$$\max_S v_{2,t} \to +\infty, \ \text{ as } t \to 0. \tag{4.28}$$

Proof. We argue by contradiction. Suppose that, along a sequence $t_n \to 0$, we have

$$0 \leq v_{2,t_n} \leq C,$$

for suitable constant $C > 0$. Then by elliptic estimates (along a subsequence), we find that $\{v_{2,t_n}\}$ converges strongly in $C^{2,\beta}(S)$ norm to $v \equiv 0$, the unique solution of problem $(1)_{t=0}$. But this is impossible, since for $t > 0$ small, the stable solution $v_{1,t}$ is the only solution of $(1)_t$ contained in a small ball centered at the origin . □

Clearly, by a similar argument, we see that any family of unstable solutions of (0.10) admits the same blow-up behavior in (4.28), as $t \to 0^+$.

5. Blow-up analysis: applications to mountain pass solutions

In this section, we apply the general blow-up analysis of §3 to the mountain pass solution $v_{2,t}$ of problem $(1)_t$ obtained in Theorem A. The asymptotic behavior of $v_{2,t}$ differs when the surface has genus two or higher.

By Proposition 4.4 and Theorem D, we know that:

$$\liminf_{t \to 0^+} \left(t^2 \int_S K(z) e^{v_{2,t}} dA \right) = 4\pi m, \tag{5.1}$$

for suitable $m \in \mathbb{N}$ satisfying $1 \leq m \leq g - 1$.

Our first goal is to prove that actually, $m = 1$. We start with the case where the genus of the surface S is at least three.

5.1. Blow-up analysis when $g \geq 3$.

Theorem 5.1. *Let the genus $g \geq 3$. Then for $K = \|\alpha\|_\sigma^2$ we have:*

(i)

$$\lim_{t\to 0} t^2 \int_S K(z) e^{v_{2,t}} dA = 4\pi. \tag{5.2}$$

(ii) *As $t \to 0$,*

$$t^2 K e^{v_{2,t}} \to 4\pi\delta_{p_0},$$

with some suitable $p_0 \in S$ such that $K(p_0) \neq 0$ (i.e. $\alpha(p_0) \neq 0$).

(iii)

$$v_{2,t} \rightharpoonup v_0 \text{ weakly in } W^{1,q}(S),\ 1 < q < 2,$$

$$v_{2,t} \to v_0 \text{ strongly in } C^{2,\beta}_{loc}(S\backslash\{p_0\}),\ 0 < \beta < 1,$$

and,

$$e^{-v_{2,t}} \to e^{-v_0} \text{ strongly in } L^p(S),\ p \geq 1.$$

Moreover, v_0 is the unique solution to the following equation on S:

$$-\Delta v_0 = 8\pi\delta_{p_0} + 2e^{-v_0} - 2.$$

In order to establish Theorem 5.1, we establish first the following estimates:

Lemma 5.2. *If the genus $g \geq 3$, then for a suitable constant $C > 0$, we have:*

$$0 \leq \fint_S v_{2,t}(z) dA \leq C, \tag{5.3}$$

and

$$|\mathcal{I}_t(v_{2,t}) - 8\pi \log \frac{1}{t^2}| \leq C, \quad \forall t \in (0, \tau_0). \tag{5.4}$$

Proof. By virtue of Corollary 3.3, clearly it suffices to prove (5.3) and (5.4) as $t \searrow 0$. We start by showing:

$$\mathcal{I}_t(v_{2,t}) \leq 8\pi \log \frac{1}{t^2} + C, \quad \text{as } t \searrow 0, \tag{5.5}$$

with a suitable constant $C > 0$ (independent of t).

To this purpose we use sharp estimates obtained in [**DJLW97**] in order to establish the existence of minimizers for the Moser-Trudinger functional $\mathcal{J}$ in (1.5). We fix $p \in S$ with $K(p) \neq 0$. As in [**DJLW97**], we use normal (polar) coordinates at p, centered at the origin, so that for $r = dist(q, p)$, we have:

$$8\pi G(r, \theta) = -4 \log r + A(p) + b_1 r \cos\theta + b_2 r \sin\theta + \beta(r, \theta), \text{ as } r \to 0,$$

with $A(p) = 8\pi\gamma(p, p)$, suitable constants b_1 and b_2 depending on the hyperbolic metric g_σ, and $\beta(r, \theta) = o(r)$ as $r \to 0$. Recall that $G(p, q)$ is the Green's function in (3.2), and $\gamma(p, q)$ its regular part.

We let η be a standard cut-off function such that:

$$\begin{cases} \eta \in C_0^\infty(B_{2a_t}(p)), \\ \eta = 1 \text{ in } B_{a_t}(p), \\ \|\nabla\eta\|_{L^\infty} \leq \frac{C}{a_t}, \end{cases}$$

where $a_t > 0$ is chosen in such a way that $a_t \to 0$ and $\alpha_t = \frac{a_t}{t} \to \infty$, as $t \to 0^+$.

Now we let,

$$\varphi_t(r,\theta) = \begin{cases} -2\log(r^2+t^2) + b_1\cos\theta + b_2\sin\theta, & \text{for } 0 \leq r \leq a_t \\ 8\pi G(r,\theta) - \eta\beta(r,\theta) - A(p) - 2\log(1+\frac{1}{\alpha_t^2}), & \text{for } a_t < r \leq 2a_t \\ 8\pi G(r,\theta) - A(p) - 2\log(1+\frac{1}{\alpha_t^2}), & \text{for } 2a_t < r. \end{cases}$$

For φ_t, we can use well-known estimates. For example from the much sharper estimates derived in [**DJLW97**] that we apply with $\epsilon = t^2$, $\phi_\epsilon = \varphi_t + \log t^2$, and $\alpha = \alpha_t$, we obtain that, as $t \to 0$,

$$\int_S |\nabla\varphi_t|^2 dA = 16\pi\log\frac{1}{t^2} - 16\pi + 8\pi A(p) + o(1), \tag{5.6}$$

$$⨍_S \varphi_t \, dA = -A(p) + o(1), \qquad \int_S e^{-\varphi_t} dA = O(1), \tag{5.7}$$

and

$$t^2 \int_S K(z)e^{\varphi_t} dA = K(p)\pi + o(1). \tag{5.8}$$

Next we construct a suitable path in $\mathcal{P}_t$ (defined in (4.27)) as follows:

$$\Gamma_t(s) = \begin{cases} (1-4s)v_{1.t}, & \text{for } 0 \leq s \leq \frac{1}{4} \\ (4s-1)\varphi_t, & \text{for } \frac{1}{4} < r \leq \frac{1}{2} \\ \varphi_t + (2s-1)\tilde{c}_t, & \text{for } \frac{1}{2} < s \leq 1, \end{cases}$$

with $\tilde{c}_t \gg 1$ sufficiently large to ensure that

$$\mathcal{I}_t(\varphi_t + \tilde{c}_t) < \mathcal{I}_t(v_{1,t}) - 10.$$

Clearly $\Gamma_t \in \mathcal{P}_t$. Furthermore, by virtue of the above estimates (5.6), (5.7), (5.8), for $t > 0$ sufficiently small, we have:

$$\begin{aligned} \max_{s\in[0,1]} \mathcal{I}_t(\Gamma_t(s)) &\leq 2\max_{c\geq 0}\{-e^c t^2 \int_S Ke^{\varphi_t} + e^{-c}\int_S e^{-\varphi_t} + 4\pi(g-1)c\} \\ &\qquad + \frac{1}{2}\|\nabla\varphi_t\|^2 + 4\pi(g-1)\int_S \varphi_t \\ &\leq 8\pi\log\frac{1}{t^2} + C, \end{aligned}$$

for some suitable $C > 0$, independent of t. In view of (4.26), this proves (5.5).

To obtain the reverse inequality, we decompose:

$$v_{2,t} = w_t + \mathsf{c}_t, \qquad \text{with } \mathsf{c}_t = ⨍_S v_{2,t}(z) dA.$$

We use the Moser-Trudinger inequality (see for instance [**Aub82**]) to estimate:

$$\begin{aligned} t^2\int_S K e^{w_t+\mathsf{c}_t} &\leq t^2 e^{\mathsf{c}_t}\|K\|_\infty \int_S e^{w_t} \\ &\leq t^2 C e^{\mathsf{c}_t} e^{\frac{\|\nabla w_t\|^2}{16\pi}}. \end{aligned} \tag{5.9}$$

By (5.1), it is necessary that:

$$\lim_{t\searrow 0} t^2\int_S K e^{v_{2,t}} \geq 4\pi, \tag{5.10}$$

and so from (5.9) we find that,

$$\|\nabla w_t\|^2 \geq 16\pi\log\frac{1}{t^2} - 16\pi\mathsf{c}_t - C_0, \tag{5.11}$$

for some suitable constant $C_0>0$.

As a consequence, we find:

$$\begin{aligned} 8\pi\log\frac{1}{t^2} + 8\pi(g-2)\mathsf{c}_t - 2t^2\int_S K e^{v_{2,t}} + 2\int_S e^{-v_{2,t}} - C_0 &\leq \mathcal{I}_t(v_{2,t}) \\ &\leq 8\pi\log\frac{1}{t^2} + C. \end{aligned} \tag{5.12}$$

We are assuming $g\geq 3$, and also we know that: $\mathsf{c}_t>0$ and $t^2\int_S K e^{v_{2,t}}dA \in (0,4\pi(g-1))$. Thus, from (5.12) we easily derive (5.3) and (5.4). □

Now we will prove Theorem 5.1:

Proof. (of Theorem 5.1) Recall that we have set, $v_{2,t} = w_t + \mathsf{c}_t$, and from (5.3) and (5.4), it follows that, as $t\searrow 0$,

$$\frac{\|\nabla w_t\|^2}{|\log t^2|} \to 16\pi. \tag{5.13}$$

While by the first estimate in (5.9) and (5.10), we also have:

$$\lim_{t\to 0}\frac{\log(\int_S e^{w_t})}{|\log t^2|} \geq 1, \text{ as } t\searrow 0. \tag{5.14}$$

As a consequence of (5.13), (5.14) and the Moser-Trudinger inequality, as $t\searrow 0$, we find:

$$\frac{\log(\int_S e^{w_t})}{\|\nabla w_t\|_2^2} \to \frac{1}{16\pi}. \tag{5.15}$$

Moreover, by (5.3) and Proposition 4.4, it follows that,

$$\max_S w_t \to +\infty, \quad \text{as } t\to 0^+. \tag{5.16}$$

By the improved Moser-Trudinger inequality of Chen-Li [**CL91**] (see Lemma 6.2.7 in [**Tar08**] and also Malchiodi-Ruiz [**MR11**]), and in view of (5.15) and (5.16), we have that, there exists a unique point $p_0\in S$, such that, $\forall r>0$ sufficiently small, the following holds as $t\searrow 0$:

$$\frac{\int_{B_r(p_0)} e^{w_t}}{\int_S e^{w_t}} \to 1, \tag{5.17}$$

$$\max_{B_r(p_0)} w_t \to +\infty, \tag{5.18}$$

and

$$\max_{S\setminus B_r(p_0)} w_t \le C_r, \tag{5.19}$$

for a suitable constant $C_r > 0$. In other words, p_0 is the unique blow-up point for w_{t_n}, along any sequence $t_n \searrow 0$, see [**BM91**]. That is, if $p_n \in S$ satisfies:

$$w_{t_n}(p_n) = \max_S w_{t_n} \to +\infty, \quad \text{as } n \to \infty$$

then:

$$p_n \to p_0, \text{ as } n \to +\infty. \tag{5.20}$$

We shall show that,

$$K(p_0) \neq 0, \tag{5.21}$$

and since $K = \|\alpha\|^2_\sigma$, p_0 cannot be a zero for $\alpha \in Q(\sigma)$.

In order to see this, we use (5.3) and (5.10) to find that,

$$\log(t^2 \int_S Ke^{w_t}) = -\mathsf{c}_t + O(1) = O(1), \text{ as } t \to 0.$$

Thus, as $t \searrow 0$, we have:

$$\begin{aligned} O(1) &= \mathcal{I}_t(v_{2,t}) + 8\pi \log \frac{1}{t^2} \\ &= \frac{1}{2}\|\nabla w_t\|^2 - 8\pi \log(\int_S Ke^{w_t}) + 8\pi(g-2)\log(t^2 \int_S Ke^{w_t}) + O(1), \end{aligned}$$

that is,

$$\frac{1}{2}\|\nabla w_t\|^2 - 8\pi \log(\int_S Ke^{w_t}) = O(1). \tag{5.22}$$

On the other hand, from (5.17) we easily check that,

$$\frac{\int_{S\setminus B_r(p_0)} Ke^{w_t}}{\int_{B_r(p_0)} e^{w_t}} \to 0, \text{ as } t \to 0, \tag{5.22a}$$

and therefore, as $t \searrow 0$:

$$\begin{aligned} \log(\int_S Ke^{w_t}) &= \log(\int_{B_r(p_0)} Ke^{w_t} + \int_{S\setminus B_r(p_0)} Ke^{w_t}) \\ &\le \log\left(\max_{B_r(p_0)}(K)\right) + \log \int_{B_r(p_0)} e^{w_t} + \log\left(1 + \frac{\int_{S\setminus B_r(p_0)} Ke^{w_t}}{\int_{B_r(p_0)} e^{w_t}}\right) \\ &< \log\left(\max_{B_r(p_0)}(K)\right) + \log \int_S e^{w_t} + o(1). \end{aligned}$$

As a consequence, from (5.22) and the Moser-Trudinger inequality, as $t \to 0^+$, we find:

$$C_1 \ge \frac{1}{2}\|\nabla w_t\|^2 - 8\pi \log(\int_S Ke^{w_t})$$

$$\begin{aligned} &\geq \frac{1}{2}\|\nabla w_t\|^2 - 8\pi \log(\int_S e^{w_t}) - 8\pi \log\left(\max_{B_r(p_0)}(K)\right) + o(1) \\ &> -C_2 - 8\pi \log \max_{B_r(p_0)}(K) + o(1). \end{aligned}$$

with suitable positive constants C_1 and C_2.

Thus, we obtain:

$$\max_{z\in B_r(p_0)}(K(z)) \geq e^{-C}, \quad \forall r > 0,$$

with a suitable constant $C > 0$, independent of $r > 0$. So by letting $r \searrow 0$, we get that $K(p_0) > 0$ and (5.21) is proved.

At this point, for any sequence $t_n \searrow 0$, we can apply Theorem D for the sequence v_{2,t_n}. In view of (5.18) and (5.19), we know that v_{2,t_n} can admit exactly <u>one</u> blow-up point at p_0 with $K(p_0) \neq 0$. Therefore (3.6) must hold with $m = 1 < g - 1$, and consequently properties (i)-(iii) must hold for v_{2,t_n}.

Since this holds along any sequence $t_n \searrow 0$, we obtain the desired conclusion. □

5.2. **Blow-up analysis when $g = 2$.** When the surface is of genus $g = 2$, the asymptotic behavior of $v_{2,t}$ is governed by the extremal properties of the Moser-Trudinger functional $\mathcal{J}$ in (1.5) (see for instance [**Tru67, Mos71, Aub82**]). Indeed, the goal of this subsection is to prove the following:

Theorem 5.3. *Let S be of genus $g = 2$. Then, as $t \searrow 0$, we have:*

$$t^2 \int_S K(z) e^{v_{2,t}} dA \to 4\pi, \tag{5.23}$$

$$\mathsf{c}_t = \fint_S v_{2,t} dA \to +\infty, \tag{5.24}$$

$$\mathcal{I}_t(v_{2,t}) - 8\pi \log \frac{1}{t^2} \to \inf_{w\in E} \mathcal{J}(w) - 8\pi, \tag{5.25}$$

where $\mathcal{J}$ is the Moser-Trudinger functional defined in (1.5)*, and $E = \{w \in H^1(S) : \int_S w(z)\ dA = 0\}$.*

Furthermore, by setting

$$w_t = v_{2,t} - \mathsf{c}_t \in E, \tag{5.26}$$

then the following alternative holds:

(i) *<u>either</u>, $\mathcal{J}$ attains its infimum on E, and along a subsequence $t = t_n \to 0$, as $n \to \infty$, we have:*

$$w_n \to w_0, \quad \text{uniformly in } C^{2,\beta}(S), \tag{5.27}$$

and

$$t^2 e^{\mathsf{c}_t} \to \frac{4\pi}{\int_S K e^{w_0}}, \tag{5.28}$$

with w_0 satisfying the following equation on S:

$$\begin{cases} -\Delta w_0 = 8\pi \left(\frac{K(z)e^{w_0}}{\int_S K(z)e^{w_0} dA} - \frac{1}{4\pi}\right) \\ \mathcal{J}(w_0) = \inf_{w\in E} \mathcal{J}(w). \end{cases} \tag{5.29}$$

(ii) *or, the functional $\mathcal{J}$ does not attain its infimum on E, and along a subsequence $t = t_n \to 0$, as $n \to \infty$, for*

$$p_n \in S \quad \text{with } w_{t_n}(p_n) = \max_S w_{t_n},$$

we have:

$$p_n \to p_0 \in S, \;\; w_{t_n}(p_n) \to +\infty, \tag{5.30}$$

and

$$t_n^2 K(z) e^{v_{2,t_n}} \to 4\pi\delta_{p_0}, \tag{5.31}$$

weakly in the sense of measure, and

$$w_{t_n} \to 4\pi G(\cdot, p_0), \tag{5.32}$$

uniformly in $C^{2,\beta}_{loc}(S\backslash\{p_0\})$, where $0 < \beta < 1$, with the blow-up point $p_0 \in S$ satisfying:

$$4\pi\gamma(p_0, p_0) + \log K(p_0) = \max_{p\in S}\{4\pi\gamma(p,p) + \log K(p)\}, \tag{5.33}$$

and in particular, $\alpha(p_0) \neq 0$.

Remark 5.4. *Clearly, if we knew the uniqueness of the minimum of the Moser-Trudinger functional $\mathcal{J}$ on E (when attained), or of the maximum point of the function $4\pi\gamma(p,p) + \log K(p)$, we could claim the convergence above as $t \to 0$, not only along a subsequence $t = t_n \to 0$ as $n \to \infty$.*

Concerning the existence of a global minimum of $\mathcal{J}$ in E, we briefly recall the work of Ding-Jost-Li-Wang ([**DJLW97**]) and Nolasco-Tarantello ([**NT98**]):

Lemma 5.5. *We have:*

$$\inf_{w\in E} \mathcal{J}(w) \leq -8\pi(\max_{p\in S}\{4\pi\gamma(p,p) + \log(K(p)\} + \log(2\pi(g-1)) + 1), \tag{5.34}$$

and the infimum is attained if (5.34) holds with a strict inequality.

Proof. See [**DJLW97, NT98**]. □

On the basis of Lemma 5.5, the existence of a global minimum for the extremal problem:

$$\inf_{w\in E}\{\frac{1}{2}\int_S |\nabla w|^2 \, dA - 8\pi \log(⨍_S K(z)e^w dA)\}, \tag{5.35}$$

was ensured by the authors in [**DJLW97, NT98**] under the following sufficient condition:

$$\Delta_{g_\sigma} \log K(p_0) > -\left(\frac{8\pi}{|S|_\sigma} - 2\kappa(p_0)\right) \tag{5.36}$$

with p_0 satisfying (5.33), and κ the Gauss curvature of (S, σ). See also [**Tar08**].

For our geometrical problem, we have $K = \frac{|\alpha|^2}{det(g_\sigma)}$, with g_σ the hyperbolic metric, and $\alpha \in Q(\sigma)$ a holomorphic quadratic differential on the Riemann surface (S, σ).

So for any $p_0 \in S$ with $\alpha(p_0) \neq 0$, we have $\kappa(p_0) = -1$, $|S|_\sigma = 4\pi$ (note that $g = 2$), and

$$\Delta_{g_\sigma} \log K(p_0) = -4.$$

Therefore we see that both sides of (5.35) are equal to -4, and in this sense we just "missed" to satisfy this sufficient condition (5.36).

Proof. We first apply (2.2) to the solution $v_{2,t}$ which (for $g = 2$) implies that (3.6) must hold with $m = g - 1 = 1$. Therefore we have, as $t \searrow 0$,

$$t^2 \int_S K e^{v_{2,t}} \to 4\pi, \quad \text{and} \quad \int_S e^{-v_{2,t}} \to 0.$$

As before, we write $v_{2,t} = w_t + \mathsf{c}_t$, and (by Jensen's inequality) we find:

$$\mathsf{c}_t \to +\infty \ \text{ as } t \searrow 0.$$

This establishes (5.23) and (5.24). Notice that we are now in the situation described by part (ii) of Theorem D.

In order to establish (5.25), we use (5.23) and $g = 2$, to conclude that the mean value c_t of $v_{2,t}$ must satisfy (2.4) with the "plus" sign. In other words,

$$e^{\mathsf{c}_t} = \frac{2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^{w_t} dA)(\int_S e^{-w_t} dA)}}{t^2 \int_S K e^{w_t} dA}. \tag{5.37}$$

So, we can use (5.37) to write

$$\begin{aligned}
\mathcal{I}_t(v_{2,t}) &= \frac{1}{2}\|\nabla w_t\|_2^2 + 8\pi \log \frac{1}{t^2} - 8\pi \log ⨍_S K e^{w_t} + 8\pi \\
&\quad +8\pi \log\Big(\frac{2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^{w_t})(\int_S e^{-w_t})}}{4\pi}\Big) - 4t^2 \int_S K e^{w_t + \mathsf{c}_t} \\
&= \frac{1}{2}\|\nabla w_t\|_2^2 + 8\pi \log \frac{1}{t^2} - 8\pi \log ⨍_S K e^{w_t} \\
&\quad +8\pi \log\Big(\frac{2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^{w_t})(\int_S e^{-w_t})}}{4\pi}\Big) \\
&\quad -4\Big(2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^{w_t})(\int_S e^{-w_t})}\Big) + 8\pi.
\end{aligned} \tag{5.38}$$

Consequently,

$$\mathcal{I}_t(v_{2,t}) - 8\pi \log \frac{1}{t^2} = \frac{1}{2}\|\nabla w_t\|_2^2 - 8\pi \log ⨍_S K e^{w_t} - 8\pi + o(1), \quad \text{as } t \to 0, \tag{5.39}$$

and we derive the lower bound:

$$\lim_{t \to 0} \left(\mathcal{I}_t(v_{2,t}) - 8\pi \log \frac{1}{t^2} \right) \geq \inf_{w \in E} \mathcal{J}(w) - 8\pi. \tag{5.40}$$

To obtain the reversed inequality, we will construct some "optimal" path. To this purpose, for any fixed $w \in E$, we find $t_w > 0$ sufficiently small, such that:

$$(t^2 \int_S K e^{w_t})(\int_S e^{-w_t}) < 4\pi^2, \quad \forall\, t \in (0, t_w).$$

So for every $t \in (0, t_w)$, we can define

$$\mathsf{c}_t^{\pm}(w) = \log \left(\frac{2\pi \pm \sqrt{(2\pi)^2 - t^2 \int_S K(z) e^w dA \int_S e^{-w} dA}}{t^2 \int_S K(z) e^w dA} \right). \tag{5.41}$$

Also set, corresponding to the stable solution $v_{1,t}$:

$$\mathsf{c}_{1,t} = \fint_S v_{1,t} \to 0, \quad \text{as } t \searrow 0, \tag{5.42}$$

and

$$w_{1,t} = v_{1,t} - \mathsf{c}_{1,t} \to 0, \quad \text{strongly in } C^{2,\beta}(S), \text{ as } t \searrow 0. \tag{5.43}$$

We define the following path:

$$\Gamma_{t,w}(s) = \begin{cases} v_{1.t} - 4s w_{1,t}, & \text{for } 0 \le s \le \frac{1}{4} \\ (4s-1)(w + \mathsf{c}_t^-(w)) + 2(1-2s)\mathsf{c}_{1,t}, & \text{for } \frac{1}{4} < s \le \frac{1}{2} \\ w + \mathsf{c}_t^-(w) + (2s-1)\tilde{C}_t, & \text{for } \frac{1}{2} < s \le 1, \end{cases} \tag{5.44}$$

with $\tilde{C}_t > 0$ fixed sufficiently large (depending on w), to ensure that,

$$\mathcal{I}_t(w + \mathsf{c}_t^-(w) + \tilde{C}_t) < \mathcal{I}_t(v_{1,t}) - 10, \quad \forall\, t \in (0, t_w).$$

Therefore $\Gamma_{t,w} \in \mathcal{P}_t$, the path space defined in (4.27). Since

$$\mathsf{c}_w^- \to \log \fint_S e^{-w}, \quad \text{as } t \searrow 0,$$

we readily check that,

$$\mathcal{I}_t(\Gamma_{t,w}(s)) \le C(w), \quad \text{for } s \in [0, \frac{1}{2}], \quad t \in (0, t_w), \tag{5.45}$$

with a suitable constant $C(w) > 0$ depending on w only.

On the other hand, for $s \in [\frac{1}{2}, 1]$ and $t \in (0, t_w)$, we have:

$$\begin{aligned} \mathcal{I}_t(\Gamma_{t,w}(s)) &\le \frac{1}{2}\|\nabla w_t\|_2^2 + 2 \max_{c \ge \mathsf{c}_w^-} \{-t^2 e^c \int_S K e^w + e^{-c} \int_S e^{-w} + 4\pi c\} \\ &= \frac{1}{2}\|\nabla w_t\|_2^2 - 2t^2 e^{\mathsf{c}_t^+(w)} \int_S K e^w + 2e^{-\mathsf{c}_t^+(w)} \int_S e^{-w} + 8\pi \mathsf{c}_t^+(w). \end{aligned}$$

So, by observing that $v = w + \mathsf{c}_t^+(w)$ satisfies the integral identity (2.2), we can use (5.41) to show that

$$\begin{aligned} \max_{s \in [\frac{1}{2}, 1]} \mathcal{I}_t(\Gamma_{t,w}(s)) \le\ & \frac{1}{2}\|\nabla w\|_2^2 + 8\pi \log \frac{1}{t^2} - 8\pi \log \fint_S K e^w \\ & + 8\pi \log \left(\frac{2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^w)(\int_S e^{-w})}}{4\pi} \right) + 8\pi \\ & - 4 \left(2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^w)(\int_S e^{-w})} \right). \end{aligned} \tag{5.46}$$

So from (5.45) and (5.46), for $t > 0$ sufficiently small, we find:

$$
\begin{aligned}
\mathcal{I}_t(v_{2,t}) - 8\pi \log \frac{1}{t^2} &\leq \max_{s\in[0,1]} \mathcal{I}_t(\Gamma_{t,w}(s)) - 8\pi \log \frac{1}{t^2} \\
&\leq \frac{1}{2}\|\nabla w\|_2^2 - 8\pi \log \fint_S K e^w + 8\pi \\
&\quad + 8\pi \log \left(\frac{2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^w)(\int_S e^{-w})}}{4\pi} \right) \\
&\quad - 4 \left(2\pi + \sqrt{4\pi^2 - (t^2 \int_S K e^w)(\int_S e^{-w})} \right).
\end{aligned} \tag{5.47}
$$

As a consequence, we get:

$$
\overline{\lim_{t\searrow 0}} \left(\mathcal{I}_t(v_{2,t}) - 8\pi \log \frac{1}{t^2} \right) \leq \frac{1}{2}\|\nabla w\|_2^2 - 8\pi \log \fint_S K e^w - 8\pi. \tag{5.48}
$$

Since (5.48) holds for every $w \in E$, and using (5.40), we establish (5.25). Actually, from (5.39) and (5.48), we see that,

$$
\lim_{t\searrow 0} \left(\frac{1}{2}\|\nabla w_t\|_2^2 - 8\pi \log \fint_S K e^{w_t} \right) = \inf_E \mathcal{J}. \tag{5.49}
$$

Next we wish to show that w_t satisfies the "compactness" alternative in part (ii) of Theorem D (along a subsequence $t = t_n \searrow 0$) if and only if $\mathcal{J}$ attains its infimum in E.

To this purpose, we fix $w \in E$, and as before set $t_w > 0$ sufficiently small to ensure that,

$$
A_t(w) = (t^2 \int_S K e^{w_t})(\int_S e^{-w_t}) < 4\pi^2, \quad \forall\ t \in (0, t_w).
$$

We set a function

$$
f(A) = \log \left(\frac{2\pi + \sqrt{4\pi^2 - A}}{4\pi} \right) + 8\pi - 4(2\pi + \sqrt{4\pi^2 - A}), \tag{5.50}
$$

for $A \in [0, 4\pi^2]$. Clearly this is a monotone increasing function of A in $[0, 4\pi^2]$.

From (5.47) and (5.38), it follows that, for $\forall w \in E$, and $\forall t \in (0, t_w)$, there holds:

$$
\frac{1}{2}\|\nabla w_t\|_2^2 - 8\pi \log \fint_S K e^{w_t} + f(A_t(w_t)) \leq \frac{1}{2}\|\nabla w\|_2^2 - 8\pi \log \fint_S K e^w + f(A_t(w)). \tag{5.51}
$$

Therefore if we assume the functional $\mathcal{J}$ attains its infimum at w_0, that is,

$$
\mathcal{J}(w_0) = \inf_E \mathcal{J}, \tag{5.52}
$$

then we can use $w = w_0$ in (5.51) to conclude that,

$$
f(A_t(w_t)) \leq f(A_t(w_0)), \quad \forall\ t \in (0, t_{w_0}). \tag{5.53}
$$

Now we use the fact that f is increasing in A and from (5.53) to derive that,

$$
(\fint_S K e^{w_t})(\fint_S e^{-w_t}) \leq (\fint_S K e^{w_0})(\fint_S e^{-w_0}),
$$

with $\fint_S e^{-w_t} \geq 1$, by Jensen's inequality.

Therefore, for suitable $C_1 > 0$, we have,

$$\fint_S Ke^{w_t} \leq C_1, \quad \forall\ t \in (0, t_{w_0}). \tag{5.54}$$

Using $v_{2,t} = w_t + \mathsf{c}_t$ in Lemma 2.4, we find that,

$$\fint_S Ke^{w_t} \geq C_2, \tag{5.55}$$

for suitable $C_2 > 0$, and moreover w_t is uniformly bounded in $W^{1,q}(S)$, $1 < q < 2$. Now from (5.37), (5.54) and (5.55), we can deduce that:

$$\frac{1}{C_3} \leq t^2 e^{\mathsf{c}_t} \leq C_3, \quad \forall\ t \in (0, t_{w_0}), \tag{5.56}$$

with a suitable constant $C_3 > 1$.

In addition, from (5.49), with $w = w_0$, we get

$$\begin{aligned} \frac{1}{2}\|\nabla w_t\|_2^2 &\leq 8\pi \log \fint_S Ke^{w_t} + \inf_E \mathcal{J} + f(A_t(w_t)) - f(A_t(w_0)) \\ &\leq C_4, \quad \forall\ t \in (0, t_{w_0}), \end{aligned} \tag{5.57}$$

with a suitable constant $C_4 > 0$.

So in case the functional $\mathcal{J}$ attains its infimum in E, then we can use the estimates in (5.56) and (5.57) together with elliptic estimates and well known regularity theory, to conclude that, w_t is uniformly bounded in $C^{2,\beta}(S)$-norm, with $0 < \beta < 1$, for any $t \in (0, t_{w_0})$. Consequently, along a subsequence $t_n \searrow 0$, $w_n := w_{t_n}$ satisfies the "compactness" property of part (ii) in Theorem D. In other words, (5.27) holds with w_0 satisfying (5.28) and (5.29).

Next suppose the functional $\mathcal{J}$ does not attain its infimum in E. Therefore, w_t can not satisfy the "compactness" property in part (ii) in Theorem D. Consequently, by (5.23) we know that, along a sequence $t_n \searrow 0$, the sequence $w_n := w_{t_n}$ must admit one ($m = 1$) blow-up point $p_0 \in S$, satisfying (5.30), (5.31), and (5.32). So we are left to show that (5.33) holds. To this purpose, from (5.49), we know that w_n defines a minimizing sequence for $\mathcal{J}$ in E and $\max_S w_n \to +\infty$, i.e., blow-up occurs. Therefore, we can use for w_n, the estimates detailed in [**DJLW97**] and [**NT98**] for any blow-up minimizing sequences of $\mathcal{J}$, to show that,

$$\begin{aligned} \inf_E \mathcal{J} &= \lim_{n\to+\infty} \frac{1}{2}\|\nabla w_n\|_2^2 - 8\pi \log \fint_S Ke^{w_n} \\ &\geq -8\pi \left(4\pi\gamma(p_0, p_0) + \log K(p_0) + \log \frac{\pi}{|S|} + 1\right). \end{aligned} \tag{5.58}$$

On the other hand, when $\mathcal{J}$ does not attain its infimum, we also know that,

$$\inf_E \mathcal{J} = -8\pi \max_{p\in S} \left(4\pi\gamma(p, p) + \log K(p) + \log \pi + 1\right). \tag{5.59}$$

See Lemma 5.5 and [**DJLW97**]. Now (5.33) follows immediately from (5.58) and (5.59). $\square$

6. Prescribing extrinsic curvature

In this section, we wish to investigate the possibility to obtain a minimal immersion of S into a hyperbolic three-manifold with prescribed total extrinsic curvature. Namely, for given $\rho \in (0, 4\pi(g-1))$, we require that for the induced metric g_0 we have:

$$\rho = \int_S (det_{g_0} \Pi) dA_{g_0}. \tag{6.1}$$

6.1. Main result and three lemmata.

Our main result is the following:

Theorem E. *Fixing a conformal structure $\sigma \in T_g(S)$, and a holomorphic quadratic differential $\alpha \in Q(\sigma)$, and $\rho \in (0, 4\pi(g-1)) \backslash \{4\pi m, m = 2, \cdots, g-2\}$, there exists a constant $t_\rho \in (0, \tau_0]$ ($\tau_0 = \tau_0(\sigma, \alpha) > 0$ given in Theorem 0.1), such that S admits a minimal immersion of data $(\sigma, t_\rho \alpha)$ into some hyperbolic three-manifold, with corresponding total extrinsic curvature satisfying* (6.1).

In order to establish this result, we need to provide a solution v_ρ for the problem $(1)_{t_\rho}$, for some $t_\rho \in (0, \tau_0]$ satisfying:

$$t_\rho^2 \int_S K(z) e^{v_\rho} \, dA = \rho, \quad K = \|\alpha\|_\sigma^2 = \frac{|\alpha|^2}{det(g_\sigma)}. \tag{6.2}$$

To this purpose, we recall from §3.1 the Mean Field formulation of the problem $(1)_t$, as described in Lemma 3.1. Then for given $\rho \in (0, 4\pi(g-1))$ we need to find a solution w of the equation (3.4), that is:

$$\begin{cases} -\Delta w = 2\rho \left(\frac{K(z)e^w}{\int_S K(z) e^w dA} - \frac{1}{|S|} \right) + 2(4\pi(g-1) - \rho) \left(\frac{e^{-w}}{\int_S e^{-w} dA} - \frac{1}{|S|} \right) \\ \int_S w(z) dA = 0. \end{cases}$$

We call this equation, (namely (3.4)), the problem $(3)_\rho$.

To this end, we start with the following observation:

Lemma 6.1. *If w solves the problem $(3)_\rho$ with $\rho \in (0, 4\pi(g-1))$, then:*

$$\left\| \frac{e^{-w}}{\int_S e^{-w} dA} - \frac{1}{|S|} \right\|_{L^\infty} < \frac{1}{4\pi(g-1) - \rho}. \tag{6.3}$$

Proof. We recall from Lemma 3.1 that when w solves problem $(3)_\rho$, then we define:

$$\mathsf{c}_\rho = \log \left(\frac{\int_S e^{-w} dA}{4\pi(g-1) - \rho} \right), \tag{6.4}$$

and

$$t_\rho^2 = \frac{\rho(4\pi(g-1) - \rho)}{(\int_S K(z) e^w dA)(\int_S e^{-w} dA)}, \tag{6.5}$$

and we see that $v_\rho = w + \mathsf{c}_\rho$ is a solution to problem $(1)_{t_\rho}$. Since we have:

$$(4\pi(g-1) - \rho) \left(\frac{e^{-w}}{\int_S e^{-w} dA} - \frac{1}{|S|} \right) = e^{-v_\rho} - ⨍_S e^{-v_\rho} \, dA,$$

and $v_\rho > 0$, then, $\forall\ z \in S$:

$$|e^{-v_\rho(z)} - \fint_S e^{-v_\rho}\ dA| < 1 - e^{-v_\rho}(z) < 1,$$

and this establishes (6.3). $\square$

As a consequence of Corollary 3.5, we know that, for any $\rho \in (4\pi(m-1), 4\pi m)$, $m = \{1, \cdots, g-1\}$, the Leray-Schauder degree of the Fredholm operator associated to the problem $(3)_\rho$ is well-defined, and its value only depends on m. To be more precise, we recall that the Laplace-Beltrami operator $\Delta = \Delta_{g_\sigma}$ is invertible on the space E. We denote by:

$$(\Delta|_E)^{-1} : E \to E$$

its (smooth) inverse. Thus each solution to the problem $(3)_\rho$ corresponds to a zero of the following operator:

$$F_\rho(w) = w + T^0_\rho(w) + B_\rho(w), \quad \forall\ w \in E, \tag{6.6}$$

with

$$T^0_\rho(w) = 2\rho(\Delta|_E)^{-1}\left(\frac{Ke^w}{\int_S Ke^w dA} - \frac{1}{|S|}\right), \tag{6.7}$$

and

$$B_\rho(w) = 2\left(4\pi(g-1) - \rho\right)(\Delta|_E)^{-1}\left(\frac{e^{-w}}{\int_S e^{-w} dA} - \frac{1}{|S|}\right). \tag{6.8}$$

Therefore, in view of Lemma 6.1, there exists a suitable constant $C > 0$ (independent of ρ), such that if $w \in E$ is a solution of problem $(3)_\rho$, that is $F_\rho(w) = 0$, then $\|B_\rho(w\| \le C$.

As a consequence, for any $\rho \in (4\pi(m-1), 4\pi m)$, with $m = \{1, \cdots, g-1\}$, we find a radius R_ρ sufficiently large, such that, for each $t \in [0,1]$ it is well-defined at zero the Leray-Schauder degree $d_{\rho,t}$ of the operator

$$F^t_\rho(w) = w + T^0_\rho(w) + tB_\rho(w), \tag{6.9}$$

in the ball $B_{R_\rho} = \{w \in E : \|w\| \le R_\rho\}$. Moreover, by the homotopy invariance of the Leray-Schauder degree, we have

$$d_\rho = d_{\rho,t}, \quad \forall\ t \in [0,1].$$

In particular,

$$d_\rho = d_{\rho,0}, \tag{6.10}$$

where $d_{\rho,0}$ is the Leray-Schauder degree of the operator

$$F^0_\rho(w) = w + T^0_\rho(w), \quad \forall\ w \in E,$$

whose zeroes correspond to solutions of the following problem:

$$\begin{cases} -\Delta w = 2\rho\left(\frac{K(z)e^w}{\int_S K(z)e^w dA} - \frac{1}{|S|}\right), & \text{on } S \\ \int_S w(z) dA = 0, \end{cases}$$

Actually for every $\rho \notin 4\pi\mathbb{N}$, the Leray-Schauder degree of the operator F^0_ρ has been computed by Chen-Lin ([**CL03, CL15**]), exactly when the weight function K

admits isolated zeros (which is our case) $\{q_1, \cdots, q_N\}$, each with integer multiplicity $\nu(q_j)$, $1 \leq j \leq N$. More precisely we have the following:

Lemma 6.2. *([**CL15**]) If $\nu(q_j) \in \mathbb{N}$ $\forall 1 \leq j \leq N$, and the genus of the surface is greater than zero, then $d_{\rho,0} > 0$.*

Proof. See Corollary 1.2 in [**CL15**]. $\square$

6.2. **Proof of Theorem E.** We now complete the proof of Theorem E:

Proof. Since in our case, the weight function $K(z) = \|\alpha\|_\sigma^2$ with $\alpha \in Q(\sigma)$ a holomorphic quadratic differential on S, we know that α admits isolated zeroes of integer multiplicity and total number (counting multiplicity) equal to $4(g-1)$. Thus, we can apply Lemma 6.2 together with (6.10) to conclude that, for every $\rho \in (0, 4\pi(g-1)) \backslash \{4\pi m, m = 1, \cdots, g-2\}$, the Leray-Schauder degree $d_\rho > 0$. In other words, for such range of ρ's, we know that the problem $(3)_\rho$ admits at least one solution. To complete the proof, we need to show that, when $g \geq 3$, then we have the existence of a solution for problem $(3)_\rho$ also when $\rho = 4\pi$.

To this purpose, we once again exploit the work of Chen-Lin in [**CL10, CL02**]. We take a sequence w_n of the solutions to problem $(3)_{\rho_n}$, with $4\pi m \neq \rho_n \to 4\pi m$, for some $m \in \{1, \cdots, g-2\}$. We assume that, as $n \to +\infty$, the following holds:

$$w_n \rightharpoonup w_0 \quad \text{weakly in } W^{1,q}(S), 1 < q < 2, \tag{6.11}$$

and

$$\max_S w_n \to +\infty, \tag{6.12}$$

Indeed, in case w_n was uniformly bounded in S, then by elliptic estimates (along a subsequence) it would converge to a solution to problem $(3)_{\rho=4\pi m}$, and for $m = 1$ we would obtain our solution in this way. Thus, we assume (6.12) and we want to establish a sign for the quantity $\rho_n - 4\pi m$. This delicate task has been carried out by Chen-Lin in [**CL10, CL02**] for the sequence $z_n = w_n - \zeta_n$, satisfying (3.17) and (3.18), with ζ_n defined in (3.14) and satisfying (along a subsequence):

$$\zeta_n \to \zeta_0, \quad \text{strongly in } C^{2,\beta}(S), \text{ as } n \to +\infty, \tag{6.13}$$

with ζ_0 the unique solution for:

$$\begin{cases} -\Delta\zeta_0 = 8\pi(g-m-1)\left(\frac{e^{-w_0}}{\int_S e^{-w_0}} - \frac{1}{|S|}\right) & \text{on } S, \\ \int_S \zeta_0(z)dA = 0, \end{cases} \tag{6.14}$$

From (6.12) we know that, $\max\limits_S z_n \to +\infty$. Therefore, by using Theorem 3.2, z_n must admit a finite number of blow-up points, say $\{p_1, \cdots, p_s\} \subset S$, for which (3.20) holds with $m = \sum\limits_{j=1}^{s}(1 + n(p_j))$ and

$$w_0(z) = \zeta_0(z) + 8\pi \sum_{j=1}^{s}(1 + n(p_j))G(z, p_j).$$

If we further assume that these blow-up points are not zeroes of the weight function $K = \|\alpha\|_\sigma^2$, that is $\alpha(p_j) \neq 0$, $\forall j = 1, \cdots, s$; then $n(p_j) = 0$ and $m = s$.

In this situation, Chen-Lin in [**CL15**] were able to control the exact decay to zero of the quantity: $\rho_n - 4\pi m$. In particular they showed that, the sign of $\rho_n - 4\pi m$ is the same of the following quantity:

$$\sum_{j=1}^{m} d_j \left(\Delta \log h^*(p_j) + \frac{8\pi m}{|S|} - 2\kappa(p_j) \right), \tag{6.15}$$

where d_j's are suitable (positive) constants, $h^* = Ke^{\zeta_0}$, and κ is the Gauss curvature of S. Take into account also that the expression (6.15) was given in [**CL15**] by formulae (2.3) and (2.10), written under the normalization $|S| = 1$. Now, for $p \in S : \alpha(p) \neq 0$, by means of (6.14), we compute:

$$\begin{aligned} \Delta \log(Ke^{\zeta_0})(p) + \frac{8\pi m}{|S|} - 2\kappa(p) &= \Delta \log \|\alpha\|_\sigma^2 + \Delta\zeta_0 + \frac{2m}{g-1} + 2 \\ &= -4 - 8\pi(g-m-1)\left(\frac{e^{-w_0}}{\int_S e^{-w_0}} - \frac{1}{4\pi(g-1)} \right) \\ &\quad + \frac{2m}{g-1} + 2 \\ &= -8\pi(g-m-1)\frac{e^{-w_0}}{\int_S e^{-w_0}} \\ &< 0, \qquad \forall\, m = \{1, \cdots, g-2\}. \end{aligned}$$

Therefore, we may conclude that, if K (and hence α) does not vanish at the blow-up points of w_n, then for n sufficiently large, we have: $\rho_n - 4\pi m < 0$. That is, blow-up can only occur from the "right".

Since for $m = 1$, the solutions to problem $(3)_{\rho_n}$ with $\rho_n \to 4\pi$ can admit only <u>one</u> blow-up point $p_0 \in S$ which must satisfy $K(p_0) \neq 0$. Therefore, we can use the information above, to see that for $\rho_n > 4\pi$ and $\rho_n \searrow 4\pi$, the corresponding solution w_n cannot blow-up, and so (along a subsequence) it converges to the desired solution of $(3)_{\rho=4\pi}$. ☐

We conclude the section with two remarks.

Remark 6.3. *When the sequence of solutions w_n to problem $(3)_{\rho_n}$ with $\rho_n \to 4\pi m$, blows up at a zero of $K = \|\alpha\|_\sigma^2$, we suspect that similar information about the sign of the quantity $\rho_n - 4\pi m$ should hold. This is confirmed by the more involved analysis developed in* [**CL10**]*, where the authors provide sharp estimates about the behavior of the sequence z_n of* (3.18), (3.17)*, which blows up at a <u>zero</u> of the weight function K, but only when such a zero is of <u>non-integer</u> multiplicity.*

Remark 6.4. *Finally we note that, in view of* (3.7), (3.8)*, by choosing $\alpha \in Q(\sigma)$ with zeroes of multiplicity greater than $g-2$, we can always guarantee that blow-up never occurs at its zeroes.*

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The Graduate Center, The City University of New York, 365 Fifth Ave., New York, NY 10016, USA

Department of Mathematics, The City University of New York, Staten Island, NY 10314, USA.

E-mail address: `zheng.huang@csi.cuny.edu`

Department of Mathematics, The City University of New York, Staten Island, NY 10314, USA.

E-mail address: `marcello.lucia@csi.cuny.edu`

Dipartimento di Matematica, Universita' di Roma "Tor Vergata", Via della Ricerca Scientifica, I-00133 Roma, ITALY.

E-mail address: `tarantel@mat.uniroma2.it`