Maximality Principles for Closed Forcings

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This is the first of a two-part talk on closed maximality principles.
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I gave the second part last week at the First European Set Theory meeting in Bedlewo, and I apologize to those who attended that talk for some overlaps between the talks.
Let’s view the universe and its possible generic extensions as a Kripke model for modal logic.
Question:

CH?
Question: \( \omega_1 > (\omega_1)^L \)?
$\phi$ is forceably necessary.

MP says $\phi$ is true.
“$\phi$ is necessary” is forceably necessary.

MP says $\phi$ is necessary.
Write $\Diamond \varphi$ to express that $\varphi$ holds in a forcing extension ($\varphi$ is forceable).

Note: This is the first order statement $\exists \mathcal{P} \quad \mathcal{P} \vDash \varphi$. 
Write $\diamond \varphi$ to express that $\varphi$ holds in a forcing extension ($\varphi$ is forceable).

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$\Box \varphi$ means that $\varphi$ holds in every forcing extension ($\varphi$ is necessary).

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Write $\Diamond \varphi$ to express that $\varphi$ holds in a forcing extension ($\varphi$ is forceable).

Note: This is the first order statement $\exists P \ P \models \varphi$.

$\Box \varphi$ means that $\varphi$ holds in every forcing extension ($\varphi$ is necessary).

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So the statement $\Diamond (\Box \varphi)$ makes sense.

It expresses that it is forceable that $\varphi$ is necessary, or in short, that $\varphi$ is forceably necessary.
Write $\diamond \varphi$ to express that $\varphi$ holds in a forcing extension (\(\varphi\) is forceable).

Note: This is the first order statement $\exists P \ P \vDash \varphi$.

$\square \varphi$ means that $\varphi$ holds in every forcing extension (\(\varphi\) is necessary).

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The Maximality Principle MP is the scheme consisting of the formulae

\[
(\diamond \square \varphi) \implies \varphi,
\]

for every sentence $\varphi$. It was introduced in a slightly different formulation in 1977 here at the Logic Colloquium by Stavi and Väänänen, and then rediscovered independently by Hamkins, as stated.
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General form of the principle:

$$\text{MP}_\Gamma(X),$$

where $\Gamma$ is a class of partial orders and $X$ is the parameter set.
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3. the class of all forcings of the form $\text{Col}(\kappa, \lambda)$ or $\text{Col}(\kappa, < \lambda)$, for some $\lambda$. Call the class $\text{Col}(\kappa)$.
I looked at the case where $\Gamma$ is one of the following, for some fixed regular cardinal $\kappa$.

1. The class of all $<_\kappa$-closed forcings,

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Note: $\kappa = \omega$ is allowed!

The corresponding parameter set will usually be one of the following:

$$\emptyset, \ H_\kappa \cup \{\kappa\}, \ H_\kappa^+.$$
1. Relationships between versions of the maximality principles.
Overview

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2. Consistency Investigations:
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The last two points were already covered in the second part of the talk.
Note the following folkloristic fact:

**Lemma 1.** Let $\kappa$ be a regular cardinal and $\lambda > \kappa$ a cardinal with $\lambda = \lambda^{<\kappa}$. Then there is a dense subset $\Delta$ of $\text{Col}(\kappa, \lambda)$ such that if $\mathbb{P}$ is a separative $<\kappa$-closed partial order with $\mathbb{P}[\lambda] = \lambda$ and $1 \Vdash_{\mathbb{P}} (\lambda = \kappa)$, then there is a dense subset $D$ of $\mathbb{P}$ with $\text{Col}(\kappa, \lambda) \upharpoonright \Delta \cong \mathbb{P} \upharpoonright D$, i.e., $\text{Col}(\kappa, \lambda)$ and $\mathbb{P}$ are forcing-equivalent.
Relationships between versions of the maximality principles

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**Corollary 2.** Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where $\kappa$ is regular. Then if $\lambda \geq \mathbb{P}$ and $\lambda^{<\kappa} = \lambda$,

$$\left(\mathbb{P} \times \text{Col}(\kappa, \lambda)\right) \upharpoonright D \cong \text{Col}(\kappa, \lambda) \upharpoonright \Delta,$$

for some dense set $D$ and the dense set $\Delta$ from Lemma 1.

So $\text{Col}(\kappa)$ absorbs any $<\kappa$-closed forcing.
Lemma 3.

\[
\text{ZFC} + \text{MP}_{\text{Col}(\kappa)}(X) \\
\vdash \text{ZFC} + \text{MP}_{\kappa-\text{dir. cl.}}(X) \\
\vdash \text{ZFC} + \text{MP}_{\kappa-\text{closed}}(X).
\]
Lemma 3.

\[ \text{ZFC} + \text{MP}_{\text{Col}(\kappa)}(X) \]
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Proof. Let \( \varphi \) be a statement with parameters from \( X \).
Lemma 3.

\[ ZFC + MP_{\text{Col}(\kappa)}(X) \]
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it suffices to show:
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Proof. Let \( \varphi \) be a statement with parameters from \( X \). To show

\[ MP_{\prec \kappa - \text{dir. cl.}}(X) \implies MP_{\prec \kappa - \text{closed}}(X), \]

it suffices to show:

\[ \varphi \text{ is } \prec \kappa - \text{closed-forceably necessary} \]

\[ \implies \]

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This can be seen as follows:

• Let $\mathbb{P}$ be a $\langle \kappa \rangle$-closed poset making $\varphi$ $\langle \kappa \rangle$-closed-necessary.

• $\mathbb{P}$ forces that it is $\langle \kappa \rangle$-closed-necessary that $\varphi$ is $\langle \kappa \rangle$-closed-necessary.

• Let $\mathbb{Q} = \text{Col}(\kappa, \theta)$, where $\theta$ is sufficiently closed and large.

• Note: $\mathbb{Q} = \text{Col}(\kappa, \theta)^{V_{\mathbb{P}}}$.

• $\varphi$ is $\langle \kappa \rangle$-closed-necessary in $V_{\mathbb{P} \times \mathbb{Q}}$.

• $\mathbb{P} \times \mathbb{Q}$ is forcing equivalent to $\mathbb{Q}$.

• $\mathbb{Q}$ is $\langle \kappa \rangle$-directed-closed.

The other statement is proven analogously.
\[ \text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\}) \leftrightarrow \text{MP}_{\text{Col}(\kappa)}(H_\kappa^+) \]

\[ \text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\}) \leftrightarrow \text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa^+) \]

\[ \text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\}) \leftrightarrow \text{MP}_{<\kappa-\text{closed}}(H_\kappa^+) \]
Theorem 4. Assume $\kappa < \delta$, $V_\delta \prec V$ and $\kappa$, as well as $\delta$, are regular. Then $\text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+})$ holds in $V[G]$, where $G$ is $V$-generic for $\mathbb{P} = \text{Col}(\kappa, < \delta)$. 
Implications
Lemma 5. Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where $\kappa$ is regular, and let $G$ be $\mathbb{P}$-generic over $V$. 
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3. If $S$ and $T$ are normal $\kappa$-trees s.t. $\text{Iso}(S, T)$ has cardinality less than $2^\kappa$, then $\text{Iso}(S, T) = (\text{Iso}(S, T))^{V[G]}$. 
Lemma 5. Let $\mathbb{P}$ be a $<_\kappa$-closed notion of forcing, where $\kappa$ is regular, and let $G$ be $\mathbb{P}$-generic over $V$.

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4. If $\varphi$ is a $\Sigma^1_1$-sentence and $A \subseteq \kappa$, then

$$\langle \kappa, <, A \rangle \models \varphi \iff (\langle \kappa, <, A \rangle \models \varphi)^{V[G]}.$$

Note that this remains true even for $\Sigma^1_2$-sentences, if $\kappa = \omega$, by Shoenfield absoluteness.
Lemma 5. Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where $\kappa$ is regular, and let $G$ be $\mathbb{P}$-generic over $V$.

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Note that this remains true even for $\Sigma^1_2$-sentences, if $\kappa = \omega$, by Shoenfield absoluteness.

5. If $T$ is a $\kappa$-Souslin tree, then $V[G] \models "T is a $\kappa$-Souslin tree."$
Corollary 6.  The following statements, if true, are $<\kappa$-closed-necessary.
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5. $T$ is a slim $\kappa$-tree which is not Kurepa.

6. $T$ is a $\kappa$-Souslin tree.

7. $\langle \kappa, <, A \rangle \models \varphi$, where $\varphi$ is a $\Sigma^1_2$ sentence and $A$ is a subset of $\kappa^n$, for some $n < \omega$. If $\kappa = \omega$, then $\Sigma^1_2$ can be replaced by $\Sigma^1_3$. 
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1. If $\kappa > \omega$, then $\Diamond_{\kappa}$ holds.

2. If $\kappa$ is the successor of the regular cardinal $\bar{\kappa}$ and $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$, then there is a $\kappa$-Souslin tree. In particular, this is true for $\kappa = \omega_1$. 
Theorem 7. Assume \( \text{MP}_{<\kappa-\text{closed}}(S \cup \{\kappa\}) \). Then

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2. If \( \kappa \) is the successor of the regular cardinal \( \bar{\kappa} \) and \( \bar{\kappa} < \bar{\kappa} \) = \( \bar{\kappa} \), then there is a \( \kappa \)-Souslin tree. In particular, this is true for \( \kappa = \omega_1 \).

3. For any \( A \subseteq H_\kappa \) with \( A \in S \), any \( \Sigma_2^1 \)-sentence \( \varphi \) and any \( <\kappa \)-closed notion of forcing \( \mathbb{P} \), it follows that

\[
\langle H_\kappa, \in, A \rangle \models \varphi \iff 1 \Vdash_{\mathbb{P}} (\langle H_{\bar{\kappa}}, \in, \bar{A} \rangle \models \varphi).
\]

So \( <\kappa \)-closed-generic \( \Sigma_2^1 \)-absoluteness over \( H_\kappa \) holds.
Theorem 7. Assume $\text{MP}_{<\kappa-\text{closed}}(S \cup \{\kappa\})$. Then

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3. For any $A \subseteq H_\kappa$ with $A \in S$, any $\Sigma^1_2$-sentence $\varphi$ and any $<\kappa$-closed notion of forcing $\mathbb{P}$, it follows that

$\langle H_\kappa, \in, A \rangle \models \varphi \iff 1 \Vdash_{\mathbb{P}} (\langle H_{\bar{\kappa}}, \in, \bar{A} \rangle \models \varphi)$.

So $<\kappa$-closed-generic $\Sigma^1_2$-absoluteness over $H_\kappa$ holds.

In case $\kappa = \omega$, generic $\Sigma^1_3$-absoluteness in parameters from $S \cap \mathcal{P}(\omega)$ follows.
Theorem 7. Assume $\text{MP}_{<\kappa}$-closed$(S \cup \{\kappa\})$. Then

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3. For any $A \subseteq H_\kappa$ with $A \in S$, any $\Sigma^1_2$-sentence $\varphi$ and any $<\kappa$-closed notion of forcing $\mathbb{P}$, it follows that

$$\langle H_\kappa, \in, A \rangle \models \varphi \iff 1 \Vdash_\mathbb{P} (\langle H_{\bar{\kappa}}, \in, \bar{A} \rangle \models \varphi).$$

So $<\kappa$-closed-generic $\Sigma^1_2$-absoluteness over $H_\kappa$ holds.

In case $\kappa = \omega$, generic $\Sigma^1_3$-absoluteness in parameters from $S \cap \mathcal{P}(\omega)$ follows.

So if $S = H_{\kappa+}$, boldface $<\kappa$-closed-generic $\Sigma^1_2(H_\kappa)$-absoluteness follows in case $\kappa > \omega$, and boldface generic $\Sigma^1_3$-absoluteness in case $\kappa = \omega$. 
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Proof. Generic $\Sigma^1_2$-Absoluteness:
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Proof. Generic $\Sigma^1_2$-Absoluteness:

- $2^{<\kappa} = \kappa = \overline{H_\kappa}$, by $\diamondsuit_\kappa$.  

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Proof. Generic $\Sigma^1_2$-Absoluteness:

- $2^{<\kappa} = \kappa = \overline{H_\kappa}$, by $\blacklozenge_\kappa$.

- If $\psi(A) = \langle H_\kappa, \in, A \rangle \models \varphi$ holds in $V$, then this is necessary.
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6. $L_{\kappa^+} \preceq L$. So $L$ is a model of $T_{\kappa, \kappa^+}$.

**Proof.** Generic $\Sigma^1_2$-Absoluteness:

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- If $\psi(A) = \langle H_{\kappa}, \in, A \rangle \models \varphi$ holds in $V$, then this is necessary.

- If $\psi(A)$ holds in $V[G]$, then this is necessary. So $\psi(A)$ is forceably necessary, and hence true in $V$. 
No slim Kurepa tree:
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- $\text{Col}(\kappa, [T])$, yields an extension in which $T$ ceases to be Kurepa.
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- So $T$ is forceably necessarily not Kurepa.
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$L_{\kappa^+} \prec L$: 
No slim Kurepa tree: Assume $T$ were Kurepa.

- $Col(\kappa, [T])$, yields an extension in which $T$ ceases to be Kurepa.
- No branches can subsequently be added to $T$.
- So $T$ is forceably necessarily not Kurepa.

$L_{\kappa^+} \prec L$: Tarski-Vaught criterion.
Lemma 8. Let $\mathcal{M}$ be a model of ZFC + MP$_{\kappa-\text{closed}}(\{\kappa\})$. Let $\delta$ be the supremum of the ordinals that are definable over $L^\mathcal{M}$ in the parameter $\kappa$. Then $L_\delta \prec L$. 
Lemma 8. Let $M$ be a model of $\text{ZFC} + \text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$. Let $\delta$ be the supremum of the ordinals that are definable over $L^M$ in the parameter $\kappa$. Then $L_\delta \prec L$.

Proof.

• $\delta \leq (\kappa^+)^M$, by $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$. 
Lemma 8. Let $M$ be a model of $\text{ZFC} + \text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$. Let $\delta$ be the supremum of the ordinals that are definable over $L^M$ in the parameter $\kappa$. Then $L_\delta \prec L$.

Proof.

- $\delta \leq (\kappa^+)^M$, by $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$,
- then verify the Tarski-Vaught criterion.
Summarizing, we have shown:
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**Corollary 9.** *The following equiconsistencies hold:*
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1. *The theory $\text{ZFC} + \text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ is transitive model equiconsistent to*

   $\text{ZFC} + \kappa$ is regular $+ \kappa < \delta + V_\delta \prec V$,

   *locally in $\kappa$.***
Summarizing, we have shown:

**Corollary 9.** The following equiconsistencies hold:

1. The theory $\text{ZFC} + \text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ is transitive model equiconsistent to
   
   $\text{ZFC} + \kappa$ is regular $+ \kappa < \delta + V_\delta \prec V$,
   
   locally in $\kappa$.

2. The theory $\text{ZFC} + \text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+}) + \delta = \kappa^+$ is transitive model equiconsistent to the theory
   
   $\text{ZFC} + \kappa$ is regular $+ \kappa < \delta + \delta$ is inaccessible $+ V_\delta \prec V$,
   
   locally in $\kappa$ and $\delta$. 
Compatibility of the closed maximality principles at $\kappa$ with $\kappa$ being a large cardinal

Lemma 10. Let $\varphi(\kappa)$ express one of the following statements about $\kappa$: $\kappa$ is inaccessible, Mahlo, subtle, Woodin.
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Compatibility of the closed maximality principles at $\kappa$ with $\kappa$ being a large cardinal

Lemma 10. Let $\varphi(\kappa)$ express one of the following statements about $\kappa$: $\kappa$ is inaccessible, Mahlo, subtle, Woodin.

1. The theory $\text{ZFC} + \text{MP}_{<\kappa-\text{closed}}(\{\kappa\}) + \varphi(\kappa)$ is transitive model equiconsistent to “$\text{ZFC} + \kappa$ is regular + $\kappa < \delta + V_\delta \prec V + \varphi(\kappa)$”, locally in $\kappa$.

2. The theory $\text{ZFC} + \text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+}) + \delta = \kappa^+ + \varphi(\kappa)$ is transitive model equiconsistent to the theory “$\text{ZFC} + \kappa$ and $\delta$ are regular + $\kappa < \delta + V_\delta \prec V$”, locally in $\kappa$ and $\delta$. 
A weak version of the following Lemma was independently proven by Leibman.

**Lemma 11.** Suppose $\kappa$ is supercompact and $\kappa < \delta$, where $\delta$ is an inaccessible cardinal such that $V_\delta \prec V$. Then there is a forcing extension $V[G]$ of $V$ in which $\text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+})$ holds and in which $\kappa$ is still supercompact.
A weak version of the following Lemma was independently proven by Leibman.

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**Proof.**

- Force to make \( \kappa \) Laver indestructible,
A weak version of the following Lemma was independently proven by Leibman.

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**Proof.**

- Force to make \( \kappa \) Laver indestructible,
- then force \( \text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+}) \).
A related Question

What is the consistency strength of a weakly compact $\kappa$ such that $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})/\text{MP}_{<\kappa-\text{closed}}(H_\kappa^+) \text{ holds?}$
What is the consistency strength of a weakly compact $\kappa$ such that $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})/\text{MP}_{<\kappa-\text{closed}}(H_\kappa^+)$ holds?

The following is worthwhile to note in this context:

**Observation 12.** Assume $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\}) + \kappa$ is weakly compact. Then the weak compactness of $\kappa$ is indestructible under $<\kappa$-closed forcing.
What is the consistency strength of a weakly compact $\kappa$ such that $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\}) / \text{MP}_{<\kappa-\text{closed}}(H_\kappa^+) \text{ holds?}$

The following is worthwhile to note in this context:

**Observation 12.** Assume $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\}) + \kappa$ is weakly compact. Then the weak compactness of $\kappa$ is indestructible under $<\kappa$-closed forcing.

**Proof.** That $\kappa$ is weakly compact is expressed by a $\Pi^1_2$-formula over $H_\kappa$. □
A Digression: The strength of an indestructibly weakly compact cardinal
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Apter and Hamkins: If $\kappa$ is weakly compact, and its weak compactness is indestructible by $\lt \kappa$-directed-closed forcing, and this indestructibility was achieved by forcing that has a closure point below $\kappa$, then $\kappa$ was supercompact in the ground model.
A Digression: The strength of an indestructibly weakly compact cardinal

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Schimmerling and Steel: If $K$ exists and $\kappa$ is weakly compact, then $\kappa$ is weakly compact in $K$ and $\kappa^+K = \kappa^+$. 
A Digression: The strength of an indestructibly weakly compact cardinal

Apter and Hamkins: If $\kappa$ is weakly compact, and its weak compactness is indestructible by $<\kappa$-directed-closed forcing, and this indestructibility was achieved by forcing that has a closure point below $\kappa$, then $\kappa$ was supercompact in the ground model.

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Fuchs and Schindler: Obtain a non-domestic mouse.
Impossible strengthenings of $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$
Impossible strengthenings of $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in $H_\kappa^+$. 
Impossible strengthenings of $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in $H_\kappa^+$. Let $\square \text{MP}_{<\kappa-\text{closed}}(H_\kappa^+)$ be the principle stating that $\text{MP}_{<\kappa-\text{closed}}(H_\kappa^+)$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with $H_\kappa^+$ interpreted in the extension).
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Note: $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in $H_\kappa^+$. Let $\Box \text{MP}_{<\kappa-\text{closed}}(H_\kappa^+)$ be the principle stating that $\text{MP}_{<\kappa-\text{closed}}(H_\kappa^+)$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with $H_\kappa^+$ interpreted in the extension).

**Theorem 13. [Fuchs/Hamkins]** $\Box \text{MP}_{<\kappa-\text{closed}}(H_\kappa^+)$ is inconsistent with ZFC, if $\kappa > \omega$. 
Impossible strengthenings of $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in $H_\kappa+$.

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**Theorem 13.** [Fuchs/Hamkins] $\square \text{MP}_{<\kappa-\text{closed}}(H_\kappa+)$ is inconsistent with ZFC, if $\kappa > \omega$.

**Proof.** Assume ZFC$+\square \text{MP}_{<\kappa-\text{closed}}(H_\kappa+)$. 
Impossible strengthenings of $\text{MP}_{\kappa<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\text{MP}_{\kappa<\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in $H_{\kappa^+}$.

Let $\Box \text{MP}_{\kappa<\kappa-\text{closed}}(H_{\kappa^+})$ be the principle stating that $\text{MP}_{\kappa<\kappa-\text{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $\kappa$-closed forcing (with $H_{\kappa^+}$ interpreted in the extension).

**Theorem 13. [Fuchs/Hamkins]** $\Box \text{MP}_{\kappa<\kappa-\text{closed}}(H_{\kappa^+})$ is inconsistent with ZFC, if $\kappa > \omega$.

**Proof.** Assume ZFC+$\Box \text{MP}_{\kappa<\kappa-\text{closed}}(H_{\kappa^+})$. Force to add a slim $\kappa$-Kurepa tree.
Impossible strengthenings of $\text{MP}_{\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\text{MP}_{\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in $H_\kappa^+$. Let $\square \text{MP}_{\kappa-\text{closed}}(H_\kappa^+)$ be the principle stating that $\text{MP}_{\kappa-\text{closed}}(H_\kappa^+)$ holds in every forcing extension obtained by $\kappa$-closed forcing (with $H_\kappa^+$ interpreted in the extension).

**Theorem 13.** [Fuchs/Hamkins] $\square \text{MP}_{\kappa-\text{closed}}(H_\kappa^+)$ is inconsistent with ZFC, if $\kappa > \omega$.

**Proof.** Assume ZFC + $\square \text{MP}_{\kappa-\text{closed}}(H_\kappa^+)$. Force to add a slim $\kappa$-Kurepa tree. Contradiction. □

Compare this with the following:
Impossible strengthenings of $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\text{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in $H_{\kappa^+}$.

Let $\square \text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ be the principle stating that $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with $H_{\kappa^+}$ interpreted in the extension).

**Theorem 13. [Fuchs/Hamkins]** $\square \text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ is inconsistent with ZFC, if $\kappa > \omega$.

**Proof.** Assume ZFC + $\square \text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$. Force to add a slim $\kappa$-Kurepa tree. Contradiction. \qed

Compare this with the following:

**Theorem 14. [Woodin]** $\square \text{MP}(\mathbb{R})$ is consistent, assuming strong axioms of infinity.
Impossible strengthenings of $\text{MP}_{<\kappa}\text{-closed}(H_\kappa \cup \{\kappa\})$

Note: $\text{MP}_{<\kappa}\text{-closed}({\kappa})$ cannot be consistently strengthened by allowing for parameters which are not in $H_\kappa^+$. Let $\square \text{MP}_{<\kappa}\text{-closed}(H_\kappa^+)$ be the principle stating that $\text{MP}_{<\kappa}\text{-closed}(H_\kappa^+)$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with $H_\kappa^+$ interpreted in the extension).

**Theorem 13. [Fuchs/Hamkins]** $\square \text{MP}_{<\kappa}\text{-closed}(H_\kappa^+)$ is inconsistent with ZFC, if $\kappa > \omega$.

**Proof.** Assume ZFC+$\square \text{MP}_{<\kappa}\text{-closed}(H_\kappa^+)$. Force to add a slim $\kappa$-Kurepa tree. Contradiction. □

Compare this with the following:

**Theorem 14. [Woodin]** $\square \text{MP}(R)$ is consistent, assuming strong axioms of infinity.

**Theorem 15. [Hamkins/Woodin]** $\square \text{MP}_{\text{ccc}}(R)$ is equiconsistent with the existence of a weakly compact cardinal.
The same proof shows that the principle $\Box MP_{\leq \kappa - \text{dir. cl.} (H_\kappa^+)}$ is inconsistent.
The same proof shows that the principle $\square\text{MP}_{\kappa-\text{dir. cl.}}(H_{\kappa^+})$ is inconsistent.

Note that it is not the case that the stronger a principle is, the stronger its necessary form is! Indeed, the following questions arise:
The same proof shows that the principle $\square \text{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa+})$ is inconsistent.

Note that it is not the case that the stronger a principle is, the stronger its necessary form is! Indeed, the following questions arise:

**Question 16.** Is $\square \text{MP}_{\text{Col}(\kappa)}(H_{\kappa+})$ consistent?

Is $\square \text{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa+})$ consistent?
Separating the principles
Separating the principles

Recall the relationships between the principles:

\[
\begin{align*}
\text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\}) & \iff \text{MP}_{\text{Col}(\kappa)}(H_\kappa^+) \\
\text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\}) & \iff \text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa^+) \\
\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\}) & \iff \text{MP}_{<\kappa-\text{closed}}(H_\kappa^+)
\end{align*}
\]

Can any of these implications be reversed?
Producing other models of closed maximality principles
Observation 17. \( \text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\}) \), if true, is \(<\kappa\)-closed-necessary. Actually, \( \text{MP}_{<\kappa-\text{closed}}(\{a\}) \) persists to \(<\kappa\)-closed extensions, for any \( a \).

The analogous statements apply to the maximality principles for \(<\kappa\)-\text{directed}-closed forcings and forcings from \( \text{Col}(\kappa) \) as well.
Producing other models of closed maximality principles

**Observation 17.** \( \text{MP}_{< \kappa} (H_{\kappa} \cup \{ \kappa \}) \), if true, is \( < \kappa \)-closed-necessary. Actually, \( \text{MP}_{< \kappa} (\{ a \}) \) persists to \( < \kappa \)-closed extensions, for any \( a \).

The analogous statements apply to the maximality principles for \( < \kappa \)-directed-closed forcings and forcings from \( \text{Col}(\kappa) \) as well.

For the boldface versions of the maximality principles for \( < \kappa \)-closed or \( < \kappa \)-directed-closed forcing, there is the following Lemma:

**Lemma 18.** Assume \( \text{MP}_{< \kappa} (H_{\kappa^+}) \). Let \( \mathbb{P} \) be a \( < \kappa^+ \)-closed notion of forcing. If \( G \) is \( \mathbb{P} \)-generic, then in \( V[G] \), \( \text{MP}_{< \kappa} (H_{\kappa^+}) \) continues to hold. This remains true if “\( < \kappa \)-closed” is replaced with “\( < \kappa \)-directed-closed”.
Producing other models of closed maximality principles

Observation 17. \( \text{MP}_{<\kappa \text{-closed}}(H_\kappa \cup \{\kappa\}) \), if true, is \( <\kappa \text{-closed-necessary} \). Actually, \( \text{MP}_{<\kappa \text{-closed}}(\{a\}) \) persists to \( <\kappa \text{-closed extensions} \), for any \( a \).

The analogous statements apply to the maximality principles for \( <\kappa \text{-directed-closed forcings} \) and forcings from \( \text{Col}(\kappa) \) as well.

For the boldface versions of the maximality principles for \( <\kappa \text{-closed} \) or \( <\kappa \text{-directed-closed forcing} \), there is the following Lemma:

**Lemma 18.** Assume \( \text{MP}_{<\kappa \text{-closed}}(H_{\kappa^+}) \). Let \( \mathbb{P} \) be a \( <\kappa^+ \text{-closed notion of forcing} \). If \( G \) is \( \mathbb{P} \text{-generic} \), then in \( V[G] \), \( \text{MP}_{<\kappa \text{-closed}}(H_{\kappa^+}) \) continues to hold. This remains true if “\( <\kappa \text{-closed} \)” is replaced with “\( <\kappa \text{-directed-closed} \).”

Note: Why is a version of the previous lemma for \( \text{Col}(\kappa) \) and \( \text{Col}(\kappa^+) \) missing?
Producing other models of closed maximality principles

Observation 17. $\text{MP}_{<\kappa\text{-closed}}(H_\kappa \cup \{\kappa\})$, if true, is $<\kappa$-closed-necessary. Actually, $\text{MP}_{<\kappa\text{-closed}}(\{a\})$ persists to $<\kappa$-closed extensions, for any $a$.

The analogous statements apply to the maximality principles for $<\kappa$-directed-closed forcings and forcings from $\text{Col}(\kappa)$ as well.

For the boldface versions of the maximality principles for $<\kappa$-closed or $<\kappa$-directed-closed forcing, there is the following Lemma:

Lemma 18. Assume $\text{MP}_{<\kappa\text{-closed}}(H_{\kappa^+})$. Let $\mathbb{P}$ be a $<\kappa^+$-closed notion of forcing. If $G$ is $\mathbb{P}$-generic, then in $V[G]$, $\text{MP}_{<\kappa\text{-closed}}(H_{\kappa^+})$ continues to hold. This remains true if “$<\kappa$-closed” is replaced with “$<\kappa$-directed-closed”.

Note: Why is a version of the previous lemma for $\text{Col}(\kappa)$ and $\text{Col}(\kappa^+)$ missing? Because there is none.
Separating $\text{MP}^{<\kappa-\text{closed}}$ from $\text{MP}^{<\kappa-\text{dir. cl.}}$. 
Lemma 19. Assuming $\kappa$ is supercompact, $\kappa < \delta$ and $V_\delta \prec V$, there is a model in which $MP_{<\kappa} \text{closed} (H_\kappa \cup \{\kappa\})$ holds, but $MP_{<\kappa} \text{dir. cl.} (H_\kappa \cup \{\kappa\})$ does not.

If moreover $\delta$ is inaccessible, then there is a model in which $MP_{<\kappa} \text{closed} (H_{\kappa^+})$ holds, but $MP_{<\kappa} \text{dir. cl.} (H_\kappa \cup \{\kappa\})$ does not.
Proof. Focus on the boldface part.
Proof. Focus on the boldface part.

- Do the Laver preparation.
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- Do the Laver preparation.

- Force $\text{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa^+})$. Call the resulting model $M$. 
Proof. Focus on the boldface part.

- Do the Laver preparation.
- Force MP$_{<\kappa^\text{dir. cl.}}(H_{\kappa^+})$. Call the resulting model $M$.
- Force over $M$ to add a $\kappa^+$-regressive $\kappa^+$-Kurepa tree.
Proof. Focus on the boldface part.

- Do the Laver preparation.

- Force $\text{MP}_{<\kappa^\text{dir.} \cdot \text{cl.}}(H_{\kappa^+})$. Call the resulting model $M$.

- Force over $M$ to add a $\kappa^+$-regressive $\kappa^+$-Kurepa tree.

The forcing is $<\kappa^+$-closed and destroys $\kappa$'s supercompactness (König-Yoshinobu). Call the model $N$. 
Proof. Focus on the boldface part.

• Do the Laver preparation.

• Force $\text{MP}^{\text{dir. cl.}}_{<\kappa}(H_{\kappa^+})$. Call the resulting model $M$.

• Force over $M$ to add a $\kappa^+$-regressive $\kappa^+$-Kurepa tree.
  
  The forcing is $<\kappa^+$-closed and destroys $\kappa$'s supercompactness (König-Yoshinobu). Call the model $N$.

• $N$ is a model of $\text{MP}^{\text{closed}}_{<\kappa}(H_{\kappa^+})$. 
Proof. Focus on the boldface part.

• Do the Laver preparation.

• Force $\text{MP}_{\kappa^- \text{dir. cl.}}(H_{\kappa^+})$. Call the resulting model $M$.

• Force over $M$ to add a $\kappa^+$-regressive $\kappa^+$-Kurepa tree.

  The forcing is $<\kappa^+$-closed and destroys $\kappa$’s supercompactness (König-Yoshinobu). Call the model $N$.

• $N$ is a model of $\text{MP}_{\kappa^- \text{closed}}(H_{\kappa^+})$.

• $N$ is not a model of $\text{MP}_{\kappa^- \text{dir. cl.}}(\{\kappa\})$. 

$\square$
Separating $\text{MP}^{<\kappa}_{\text{dir. cl.}}$ from $\text{MP}^{\text{Col}(\kappa)}$
Lemma 20.

1. $\text{MP}_{\text{Col}(\kappa)}(\emptyset)$ implies that $V \neq \text{HOD}$. 
Separating $\text{MP}_{<\kappa-\text{dir. cl.}}$ from $\text{MP}_{\text{Col}(\kappa)}$

Lemma 20.

1. $\text{MP}_{\text{Col}(\kappa)}(\emptyset)$ implies that $V \neq \text{HOD}$.

2. $\text{MP}_{<\kappa-\text{closed}}(\emptyset)$ implies that there is a forcing extension of an initial segment of $L$ in which $\text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\}) + V = \text{HOD}$ holds. Analogously, $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ implies that there is a forcing extension of $L$ in which $\text{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa^+}) + V = \text{HOD}$ holds.
Proof. Part 1:
Proof. Part 1: "\( \forall \neq \text{HOD} \)" is \( \text{Col}(\kappa) \)-forceably necessary.
Proof. Part 1: “$V \neq \text{HOD}$” is $\text{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$. 

Part 2: Focus on the boldface claim. Let \( \delta = (\kappa^+) \).

- \( L_\delta \prec L \).
Proof. Part 1: “V \neq \text{HOD}” is Col(\kappa)-forceably necessary.

Part 2: Focus on the boldface claim. Let \( \delta = (\kappa^+) \).

- \( L_\delta \prec L \).

- Let \( G \) be \( \text{Col}(\kappa, < \delta) \)-generic over \( L \). So \( L[G] \) is a model of \( \text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+}) \).
Proof. Part 1: “$\mathcal{V} \neq \text{HOD}$” is $\text{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$. 

- $L_\delta \prec L$.

- Let $G$ be $\text{Col}(\kappa, < \delta)$-generic over $L$. So $L[G]$ is a model of $\text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+})$.

- Force to code $G$ into the continuum function well above $\delta$. 
Proof. Part 1: “$V \neq \text{HOD}$” is $\text{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$.  

- $L_\delta \prec L$.

- Let $G$ be $\text{Col}(\kappa, < \delta)$-generic over $L$. So $L[G]$ is a model of $\text{MP}_{\text{Col}(\kappa)}(H_{\kappa^+})$.

- Force to code $G$ into the continuum function well above $\delta$.

- The result is a model of $V = \text{HOD}$, where $\text{MP}_{<\kappa - \text{closed}}(H_{\kappa^+})$ still holds, because the forcing was $<\kappa^+$-closed.
Boldface vs. lightface Principles
Lemma 21.

1. Assuming $\text{MP}_{\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$, there is a forcing extension in which $\text{MP}_{\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ holds but $\text{MP}_{\kappa-\text{closed}}(H_{\kappa^+})$ fails.
Lemma 21.

1. Assuming \( \text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\}) \), there is a forcing extension in which \( \text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\}) \) holds but \( \text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+}) \) fails.

2. Assuming \( \text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\}) \), there is forcing extension in which \( \text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\}) \) holds but \( \text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+}) \) fails.
**Lemma 21.**

1. Assuming $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$, there is a forcing extension in which $\text{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ holds but $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ fails.

2. Assuming $\text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\})$, there is forcing extension in which $\text{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\})$ holds but $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ fails.

3. Assuming $\text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$, there is a model of $\text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$ in which $\text{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ is false.
Lemma 21.

1. Assuming $\text{MP}_{<\kappa} - \text{closed}(H_\kappa \cup \{\kappa\})$, there is a forcing extension in which $\text{MP}_{<\kappa} - \text{closed}(H_\kappa \cup \{\kappa\})$ holds but $\text{MP}_{<\kappa} - \text{closed}(H_{\kappa^+})$ fails.

2. Assuming $\text{MP}_{<\kappa} - \text{dir. cl.}(H_\kappa \cup \{\kappa\})$, there is forcing extension in which $\text{MP}_{<\kappa} - \text{dir. cl.}(H_\kappa \cup \{\kappa\})$ holds but $\text{MP}_{<\kappa} - \text{closed}(H_{\kappa^+})$ fails.

3. Assuming $\text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$, there is a model of $\text{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$ in which $\text{MP}_{<\kappa} - \text{closed}(H_{\kappa^+})$ is false.
So in general, none of the implications shown in the figure can be reversed.