THE STATIONARITY OF THE COLLECTION OF THE LOCALLY REGULARS

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Abstract. I analyze various natural assumptions which imply that the set $\{\omega_1^{L[x]} \mid x \subseteq \omega\}$ is stationary in ω_1. The focal questions are which implications hold between them, what their consistency strengths are, and which large cardinal assumptions outright imply them.

1. Introduction

The motivation for this paper was a question asked by Philipp Schlicht, Paweł Kawa and Daisuke Ikegami at the Young Set Theory meeting 2008 in Bonn. Recall:

Definition 1.1. ω_1 is inaccessible to reals iff $\omega_1^{L[x]} < \omega_1$, for all $x \in \mathbb{R}$.

Fact 1.2. *The following are equivalent:*

(1) ω_1 *is inaccessible to reals.*
(2) *For every* $x \subseteq \omega$, ω_1^{V} *is inaccessible in* $L[x]$.
(3) ω_1 *is inaccessible in* $L[b]$, *for every bounded subset* b *of* ω_1.

In order to formulate the initial question concisely, I want to make the following definition.

Definition 1.3. Let

$$\Omega = \{\omega_1^{L[x]} \mid x \in \mathbb{R}\} \cap \omega_1.$$

I am mostly interested in Ω in the case that ω_1 is inaccessible to reals. In that case, $\Omega = \{\omega_1^{L[x]} \mid x \in \mathbb{R}\}$. The original question was:

Question 1.4. Is Ω stationary?

To exclude trivial counterexamples, one should ask instead:

Question 1.5. If ω_1 is inaccessible to reals, then does it follow that Ω is stationary?

It is also natural to ask:

Question 1.6. If ω_1 is inaccessible to reals, then does it follow that Ω contains a club?

I will analyze this and a number of related questions and concepts. In section 2, I will show that Ω is a very natural set. It, and its stationarity, can be expressed in many different ways, particularly if ω_1 is inaccessible to reals. One description is

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that Ω is the set of countable ordinals that are regular in L, and another is that Ω is the set of countable ordinals that are of the form ω_1^M, for some inner model M.

Using this, it is very easy to see that the consistency strength of the stationarity of Ω is exactly a Mahlo cardinal. This is shown in section 3.

In section 4, I isolate a property that seems central in this context, which implies that Ω is stationary: the nonexistence of a reshaped set. I show this implication, and that the consistency strength of no reshaped set is again a Mahlo cardinal. To get the consistency, I show that $\mathsf{MA}_{\omega_1}(\sigma\text{-centered})$, together with the inaccessibility of ω_1 to reals, implies that there is no reshaped set. On the other hand, I show that the nonexistence of a reshaped set does not imply $\mathsf{MA}_{\omega_1}(\sigma\text{-centered})$. It is also independent of the projective sets having the regularity properties.

In section 5, I introduce relativized versions Ω_a of Ω, where, basically, L in the definition of Ω is replaced with $L[a]$ in the definition of Ω_a, for a real a. I introduce a strengthening of "Ω is stationary" by defining that Ω is uniformly stationary if Ω_a is stationary, for every real a. I show that the nonexistence of a reshaped set implies the uniform stationarity of Ω.

Section 6 introduces the concept of local Mahloness. Thus, ω_1 is Mahlo at $a \subseteq \omega$ if the set of countable ordinals that are inaccessible in L is stationary (in V), and ω_1 is locally Mahlo if this is true for all reals a. I show that uniform stationarity of Ω is the equivalent to local Mahloness of ω_1. Local Mahloness is a weaker form of the statement that ω_1 is Mahlo to reals, which means that ω_1 is Mahlo in $L[a]$, for every real a.

In section 7, I show that a variety of large cardinal or forcing axioms imply that there is no reshaped set, and thus, that Ω is uniformly stationary. This is further evidence that these are very natural assumptions.

Finally, in section 8, I show that no additional implications between these concepts are provable than those that I have shown so far, so figure 1 (on page 10) is complete.

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2. Characterizations of Ω

In this section, I would like to argue that Ω is a very canonical set. Particularly under the assumption that ω_1 is inaccessible to reals, it has many equivalent and very natural definitions. I will use some basic techniques from coding theory.

Definition 2.1. A set $b \subseteq \omega_1$ is *reshaped* if for all $\xi < \omega_1$, ξ is countable in $L[b\cap\xi]$, in other words, if $\omega_1^{L[b\cap\xi]} > \xi$.

A proof of the following theorem can be found in [JBW82, Thm. 1.1], see also [JS70].

Theorem 2.2 (Solovay)**.** *Assume* $\mathrm{V} = L[b]$*, where b is reshaped. Then there is a c.c.c.*[1] *forcing $\mathbb{P}_b \subseteq [\omega]^{<\omega} \times [b]^{<\omega}$ such that $\mathbb{P}_b$ adds a real x with $b \in L[x]$.*

Another useful fact on reshaped sets is:

Theorem 2.3 (Jensen, Schindler)**.** *Let $A \subseteq \omega_1$ be such that $H_{\omega_2} \subseteq L[A]$. Then there is a σ-distributive and stationary set preserving forcing $\mathbb{P}$, called the* reshaping

[1] It's even σ-centered, which will be of importance later.

forcing, *consisting of bounded subsets of* ω_1*, which adds a set* $b \subseteq \omega_1$ *that is reshaped and codes* A*, in the sense that* $A \in L[b]$.[2]

Proof. The forcing $\mathbb{P}$ consists of bounded subsets s of ω_1 such that for every ξ, $L[s \cap \xi]$ sees that ξ is countable, and such that for every limit ordinal ξ with $\xi + 2(n+1) \le \sup(s)$, $\xi + 2(n+1) \in s$ iff $\xi + n \in A$. The ordering is that of end-extension. The first condition insures that the generic subset of ω_1 will be reshaped, and the second one insures that A will be coded. A proof that this forcing is countably distributive can be found in [Sch04, Lemma 4.5], and a proof that it preserves stationary subsets of ω_1 can be found in [Sch04, Lemma 4.8]. Stationarity preservation was originally shown in [Sch01]. $\square$

Definition 2.4. Let

$$\bar{\Omega} = \{\omega_1^{L[b]} \mid L[b] \models \text{“}b \text{ is reshaped”}\} \cap \omega_1$$

and set

$$\bar{\Omega}' = \{\omega_1^{L[c]} \mid c \subseteq \mathrm{On}\} \cap \omega_1$$

Observation 2.5. $\Omega \subseteq \bar{\Omega} \subseteq \bar{\Omega}'$.

Proof. The second inclusion is trivial, and so is the first: if $x \subseteq \omega$, then $L[x] \models$ x is reshaped, so if $\omega_1^{L[x]} < \omega_1$, then $\omega_1^{L[x]} \in \bar{\Omega}$. $\square$

If ω_1 is inaccessible to reals, then the converse is also true:

Theorem 2.6. *If* ω_1 *is inaccessible to reals, then* $\bar{\Omega}' \subseteq \bar{\Omega} \subseteq \Omega$.

Proof. For the first inclusion, let $\rho = \omega_1^{L[c]} \in \bar{\Omega}'$, for some $c \subseteq \mathrm{On}$. Working inside $L[c]$, for every $\alpha < \omega_1$ (that is, $\omega_1^{L[c]}$), let $f^\alpha : \omega \longrightarrow \alpha$ be a surjection. Let $c' = \{\prec\alpha, m, f^\alpha(m)\succ \mid \alpha < \omega_1^{L[c]} \text{ and } m < \omega\}\}$. Clearly then, $c' \subseteq \omega_1^{L[c]} = \omega_1^{L[c']} = \rho$. Now, in $L[c']$, theorem 2.3 can be applied (with $A = c'$), so the reshaping forcing $\mathbb{P}$ with respect to c' is σ-distributive. In V, since ω_1 is inaccessible to reals, there is a $b \subseteq \omega_1^{L[c']}$ which is generic for the reshaping forcing (because there are only countably many subsets of that forcing in $L[c']$). Since b codes c' and the forcing preserves ω_1, it follows that $\rho = \omega_1^{L[b]}$, and so, $\rho \in \bar{\Omega}$.

The argument for the second inclusion is similar. Let $\rho \in \bar{\Omega}$. Let $b \subseteq \rho$ be as in the definition of $\bar{\Omega}$, so $\rho = \omega_1^{L[b]}$, where b is reshaped in $L[b]$. Note that $\rho < \omega_1$. One can now force with the Solovay's almost disjoint coding forcing $\mathbb{P} = (\mathbb{P}_b)^{L[b]}$ over $L[b]$. That forcing is contained in $[\omega]^{<\omega} \times [b]^{<\omega}$. Note that $\mathcal{P}(\mathbb{P}) \cap L[b]$ is countable in V, as ω_1^{V} is inaccessible in $L[b]$. So let x be a generic real over $L[b]$. Then since $b \in L[x]$, and $\mathbb{P}$ is c.c.c. in $L[b]$, $\omega_1^{L[x]} = \omega_1^{L[b]} = \rho \in \Omega$. $\square$

This provides the following very nice and canonical characterization of Ω, assuming ω_1 is inaccessible to reals.

Theorem 2.7. *If* ω_1 *is inaccessible to reals, then* Ω *is the collection of all countable ordinals* α *with the property that for some inner model* $M \models \mathsf{ZFC}$, $M \models \alpha = \omega_1$.

Proof. For the substantial direction, suppose M is an inner model, and $\alpha = \omega_1^M < \omega_1^{\mathrm{V}}$. Working inside M, construct a set $c \subseteq \alpha$ such that $\alpha = \omega_1^{L[c]}$, as in the proof of theorem 2.6. This shows that $\alpha \in \bar{\Omega}' = \Omega$. $\square$

[2]The reshaping forcing was introduced and shown to be ω-distributive by Jensen (see [JBW82, Thm. 1.4]). Schindler showed that it preserves stationary sets, see [Sch01, Claim 3']).

Theorem 2.8. *If ω_1 is inaccessible to reals, then Ω is the collection of all countable ordinals $\alpha > \omega$ that are regular in some inner model $M \models \mathsf{ZFC}$, and this is the same as the collection of all countable ordinals greater than ω that are regular in L.*

Proof. The second part of the claim is obvious: If α is a regular cardinal in M, some inner model of ZFC, then α is also a regular cardinal in L. So it needs to be shown that this collection is equal to Ω. I will use the characterization of Ω given by the previous theorem. Clearly, every countable ordinal of the form ω_1^M, for some $M \models \mathsf{ZFC}$, is a regular cardinal in L. For the converse, let $\alpha > \omega$ be countable, yet regular in L. If α is a limit cardinal in L, then α is inaccessible in L, and I want to let $\mathbb{P} = \mathrm{Col}(\omega, {<}\alpha)$ be the collapse of α to be ω_1 (from the point of view of L). If α is a successor cardinal in L, then let $\bar{\alpha}$ be the predecessor cardinal of α in L, and let $\mathbb{P} = \mathrm{Col}(\omega, \bar{\alpha})$. In both cases, forcing with $\mathbb{P}$ adds a subset g of α such that $\alpha = \omega_1^{L[g]}$. Since ω_1^{V} is inaccessible in L, such $\mathbb{P}$-generics exist in V, so $L[g]$ witnesses that there is an inner model in which α is the first uncountable ordinal. $\square$

So, if ω_1 is inaccessible to reals, then it makes sense to refer to Ω as the collection of the local $\aleph_1$'s, or of the locally regulars.

3. The consistency strength

The following simple observation essentially answers the initial question 1.5 in the negative:

Observation 3.1. *Suppose that Ω is stationary. Then ω_1 is Mahlo in L.*

Proof. Let $C \in L$ be club in ω_1. Pick x such that $\omega_1^{L[x]} \in C$. $\omega_1^{L[x]}$ is regular in L, so we're done. $\square$

This argument shows that if Ω is stationary, then the set of countable ordinals that are inaccessible in L is stationary in ω_1 (not only in L), because the intersection of Ω with the set of limit cardinals of L is a stationary subset.

Corollary 3.2. *From an inaccessible cardinal, it is consistent that ω_1 is inaccessible to reals and Ω is not stationary.*

Proof. Let κ be the least inaccessible cardinal in L, and consider $L[G]$, where G is $\mathrm{Col}(\omega, {<}\kappa)$-generic over L. Then in $L[G]$, ω_1 is inaccessible to reals and ω_1 is not Mahlo in L, so Ω can't be stationary. $\square$

But is it consistent that Ω is (uniformly) stationary? The key is again to collapse an inaccessible cardinal to be ω_1, except this time, it's not only inaccessible but Mahlo. The following considerably simplifies an argument due to Philipp Schlicht and myself:

Lemma 3.3. *Suppose κ is Mahlo. Let G be $\mathrm{Col}(\omega, {<}\kappa)$-generic over V. Then in $M := \mathrm{V}[G]$, $\Omega = \bar{\Omega}$ is stationary.*

Proof. Clearly, ω_1 is inaccessible to reals in M. So there are many characterizations of Ω^M. One of these is that Ω is the collection of all countable ordinals that are regular in an inner model. Let S be the set of V-regular cardinals. This set is stationary in V, and since $\mathrm{Col}(\omega, {<}\kappa)$ is ${<}\kappa$-c.c., it follows that S is still stationary in M (${<}\kappa$-c.c. forcings preserve stationary subsets of κ, because for every club subset $T \subseteq \kappa$ in the extension, there is a club subset $T' \subseteq \kappa$ in the ground model such that $T' \subseteq T$). But $S \subseteq \Omega$, since from the point of view of M, every member of S is a countable ordinal which is regular in L. $\square$

Theorem 3.4. *The consistency strength of the statement "Ω is stationary" is exactly a Mahlo cardinal.*

4. A natural strengthening

It turns out that the assumption that there is no reshaped set is a strong form of the stationarity of Ω. It also implies that ω_1 is inaccessible to reals, and hence provides a very natural setting for the analysis.

Theorem 4.1. *Assume there is no reshaped set. Then ω_1 is inaccessible to reals and Ω is stationary.*

Proof. First, observe that ω_1 is inaccessible to reals. For if $x \subseteq \omega$ were such that $\omega_1^{L[x]} = \omega_1$, then x would trivially be reshaped, so such an x does not exist under our assumption.

So by theorem 2.6, $\bar{\Omega} = \Omega$, and it suffices to show that $\bar{\Omega}$ is stationary. So let C be club in ω_1. Construct a sequence $s : \rho \longrightarrow 2$ by recursion (the size of ρ will turn out to be countable), so that the following conditions are met:

(1) if $\alpha < \rho$ is a limit ordinal, then $\omega_1^{L[s\restriction\alpha]} > \alpha$,
(2) if $\beta < \alpha < \rho$, β and α limits, then $(\omega_1^{L[s\restriction\beta]}, \omega_1^{L[s\restriction\alpha]}) \cap C \neq \emptyset$.
(3) $\omega_1^{L[s]} = \rho$.

Suppose that γ is a limit ordinal and $s\restriction\gamma$ has been defined, so that 1. and 2. above hold, with ρ replaced by γ.

If $\omega_1^{L[s\restriction\gamma]} \leq \gamma$, then set $\rho = \gamma$, so $s = s\restriction\gamma$, and the construction is done. Then 1. and 2. are satisfied, and 3. holds also: By assumption, $\omega_1^{L[s]} \leq \rho$, and for every limit ordinal $\alpha < \rho$, $\omega_1^{L[s]} \geq \omega_1^{L[s\restriction\alpha]} > \alpha$, so $\omega_1^{L[s]} \geq \rho$.

In the other case, $\delta := \omega_1^{L[s\restriction\gamma]} > \gamma$. Let $\epsilon = \min(C \setminus (\delta + 1))$, and pick $x \in {}^{\omega}2$ s.t. $\omega_1^{L[x]} > \epsilon$. Then set $s(\gamma + n) = x(n)$. So this defines $s\restriction(\gamma + \omega)$, and clearly, 1. and 2. hold even at $\alpha = \gamma + \omega$.

Since there is no reshaped set, by our assumption, the construction breaks down at a countable stage ρ (note that by 1., $\rho \leq \omega_1$. But if $\rho = \omega_1$, then $\{\mu < \omega_1 \mid s(\mu) = 1\}$ is reshaped, by 1., which is a contradiction).

Clearly, ρ is a limit point of C: Let $\beta < \rho$. By 3., $\alpha := \beta + \omega < \rho$. By 2., there is a $\xi \in (\omega_1^{L[s\restriction\beta]}, \omega_1^{L[s\restriction\alpha]}) \cap C$. By 1., $\beta < \omega_1^{L[s\restriction\beta]}$, and by 3., $\omega_1^{L[s\restriction\alpha]} < \rho$. So altogether, $\beta < \xi < \rho$, and $\xi \in C$.

So $\rho \in C$. But, letting $b := \{\alpha < \rho \mid s(\alpha) = 1\}$, it follows that b is reshaped in $L[b]$, by 1. and 3. So by 3., $\rho = \omega_1^{L[b]} \in \bar{\Omega}$. So $\rho \in \bar{\Omega} \cap C$. □

The nonexistence of a reshaped set can be formulated in a more positive way:

Observation 4.2. *There is no reshaped set iff for every $b \subseteq \omega_1$, there is an $\alpha < \omega_1$ such that $\alpha = \omega_1^{L[b\cap\alpha]}$.*

Proof. Clearly, if the second property holds, then b is not reshaped. Vice versa, suppose there is no reshaped set, and let $b \subseteq \omega_1$. Since b is not reshaped, there is a least α such that $\kappa = \omega_1^{L[b\cap\alpha]} \leq \alpha$. Suppose $\kappa < \alpha$. By minimality of α, $\omega_1^{L[b\cap\kappa]} > \kappa$, so κ is countable in $L[b\cap\kappa]$, and hence in $L[b\cap\alpha]$. This is a contradiction, so $\kappa = \alpha$, as wished. □

Clearly, a Mahlo cardinal is a lower bound on the consistency strength of the nonexistence of a reshaped set, because the latter implies the stationarity of Ω, which implies that ω_1 is Mahlo in L. The following connection to a version of Martin's Axiom will show that it is an upper bound as well.

Definition 4.3. A partial ordering $\mathbb{P}$ is σ-centered iff $\mathbb{P}$ can be partitioned into countably many subsets such that any finitely many conditions that belong to the same element of the partition are compatible, i.e., have a common strengthening in $\mathbb{P}$.

Lemma 4.4. *Assume ω_1 is inaccessible to reals and $\mathsf{MA}_{\omega_1}(\sigma$-centered$)$. Then there is no reshaped set.*

Proof. Assume b was reshaped. In V, there is then a sufficiently pseudo-generic filter for the forcing $\mathbb{P}_b$ mentioned in 2.3 (and described in [JBW82, Theorem 1.1]), by $\mathsf{MA}_{\omega_1}(\sigma$-centered): $\mathbb{P}_b$ consists of conditions $p = \langle s(p), s^*(p)\rangle$ with $s(p) \in [\omega]^{<\omega}$ and $s^* \subseteq [b]^{<\omega}$. To define the ordering of $\mathbb{P}_b$, define recursively a sequence $\langle R_\alpha \mid \alpha < \omega_1\rangle$ of reals by letting R_α be minimal in $L[b \cap \alpha]$ such that $R_\alpha \notin \{R_\beta \mid \beta < \alpha\}$. This is possible, since $\omega_1^{L[b\cap\alpha]} > \alpha$. Let R^*_α be the set of $n < \omega$ which code an initial segment of R_α. So this way, the R^*_α's are pairwise almost disjoint (in the sense of bounded intersections). The ordering is now: $p \le q$ iff $s(q) \subseteq s(p)$, $s^*(q) \subseteq s^*(p)$ and for all $\alpha \in s^*(q)$, $(s(p) \setminus s(q)) \cap R^*_\alpha = \emptyset$. So if finitely many conditions have the same first co-ordinate, they have a common extension, which shows that $\mathbb{P}_b$ is σ-centered (partitioning $\mathbb{P}_b$ into the sets of conditions which have the same first coordinate).

For $\xi \in b$, the set $D^I_\xi := \{p \in \mathbb{P}_b \mid \xi \in s^*(p)\}$ is dense. Meeting these dense sets insures that $\xi \in b \implies R \cap R^*_\xi$ is finite, where R is the pseudo generic real added by $\mathbb{P}_b$. For $\xi \notin b$ and $n < \omega$, consider the set $D^{II}_\xi := \{p \in \mathbb{P}_b \mid \overline{\overline{s(p)}} > n\}$. This is again dense in $\mathbb{P}_b$, and meeting all of these sets insures the converse of the above implication. So meeting all of these dense sets is sufficient to get a real x which codes b, and hence satisfies $\omega_1^{L[x]} = \omega_1^{\mathrm{V}}$, contradicting the assumption that ω_1 is inaccessible to reals. □

Theorem 4.5. *The following are equiconsistent over* ZFC*:*

(1) Ω *is stationary.*
(2) Ω *contains a club.*
(3) *There is no reshaped set.*
(4) *There is a Mahlo cardinal.*

Proof. (1) and (4) are already known to be equiconsistent, and I have shown that (3) implies (1) Clearly, (2) implies (1), and over a model of (1), one may shoot a club through Ω using the standard forcing which is countably distributive, to produce a model of (2). So it remains to prove the consistency of (3), given that of (4). But over a model with a Mahlo cardinal, one can force a model where ω_1 is inaccessible to reals and $\mathsf{MA}_{\omega_1}(\sigma$-centered) holds, by [IS89]. By the previous lemma, there is no reshaped set in such a model. □

It was shown in [IS89] that $\mathsf{MA}_{\omega_1}(\sigma$-centered) together with the assumption that ω_1 is inaccessible to reals is equiconsistent with $\mathsf{MA}_{\omega_1}(\sigma$-centered) together with the assertion that all projective sets of reals are Lebesgue-measurable (or are Baire, or Ramsey). Since the nonexistence of a reshaped set has the same consistency strength, it is natural to ask whether an implication holds here - recall that it implies that ω_1 is inaccessible to reals. The following lemmas show that no such implication holds.

Lemma 4.6. *The nonexistence of a reshaped set is preserved by σ-closed forcing.*

Proof. Assume that there is no reshaped set, and let $\mathbb{P}$ be a σ-closed forcing. Suppose $\dot{b}$ is a $\mathbb{P}$-name and $p_0 \in \mathbb{P}$ a condition that forces "$\dot{b}$ is reshaped." Construct a decreasing sequence $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ of conditions such that for every $\alpha < \omega_1$, p_α decides "$\check{\alpha} \in \dot{b}$", using the countable closure of $\mathbb{P}$. Let $b = \{\alpha < \omega_1 \mid p_\alpha \Vdash \text{"}\check{\alpha} \in \dot{b}\text{"}\}$. Then b is reshaped, because otherwise, if α is least such that $\omega_1^{L[b\cap\alpha]} \leq \alpha$, then, letting $G \subseteq \mathbb{P}$ be V-generic, with $p_\alpha \in G$, it follows that in V$[G]$, $\dot{b}^G$ is reshaped, since $p_0 \in G$. But $\dot{b}^G \cap \alpha = b \cap \alpha$, so $\omega_1^{L[b\cap\alpha]} > \alpha$, a contradiction. □

Lemma 4.7. *If it is consistent that there is no reshaped set, then it is consistent that there is no reshaped set and the continuum hypothesis holds.*

Proof. Suppose there is no reshaped set. The forcing to add a Cohen subset of ω_1 with countable conditions is σ-closed, does not add a reshaped set (by the previous lemma), and forces CH. □

Corollary 4.8. *If it is consistent that there is no reshaped set, then it is consistent that there is no reshaped set and* MA_{ω_1}*(σ-centered) fails. So the nonexistence of a reshaped set does not imply* MA_{ω_1}*(σ-centered).*

Proof. By the previous lemma, it is consistent that there is no reshaped set and CH holds. But then MA_{ω_1}(σ-centered) cannot hold, because this would imply that CH fails. This is because the forcing to add a Cohen real is σ-centered (it is countable). There can be no filter for that forcing which meets, simultaneously for every real r, the dense set of conditions that express that the generic real is different from r. □

Note that MA_{ω_1}(σ-centered), together with the assertion that ω_1 is inaccessible to reals, implies the nonexistence of a reshaped set and the failure of the continuum hypothesis. So the continuum hypothesis is independent of the nonexistence of a reshaped set.

So it is possible to force with a σ-closed forcing over a model in which there is no reshaped set, to produce a model where there still is no reshaped set, yet MA_{ω_1}(σ-centered) fails. But what can be said about the relationship between the nonexistence of a reshaped set and the regularity properties for the projective sets? It is known that the Lebesgue-measurability of all Σ^1_3 sets of reals implies that ω_1 is inaccessible to reals, by Shelah's celebrated [She84]. But it is clear that this cannot be strengthened to conclude the nonexistence of a reshaped set, on grounds of consistency strength: the consistency strength of the regularity properties for all projective sets is an inaccessible cardinal, while the consistency strength of the nonexistence of a reshaped set is a Mahlo cardinal. But does the nonexistence of a reshaped set imply the regularity properties of the projective sets? It is known (due to Solovay) that if ω_1 is inaccessible to reals, then every $\utilde{\Sigma}^1_2$ set of reals is Lebesgue-measurable and has the Baire property. Does the nonexistence of a reshaped set give more? The answer is again no. For example, lemma 7.4 shows that if every subset of ω_1 has a sharp, then there is no reshaped set. As a consequence, for example, in $L[U]$, the minimal model with a measurable cardinal, there is no reshaped set, and it is known that $L[U]$ has a Δ^1_3-definable well-ordering of the reals, see [Sil71]. But no well-ordering of the reals can be Lebesgue-measurable or have the Baire property. So, at least from the consistency of a measurable cardinal, it can be seen that the non-existence of a reshaped set does not imply the regularity properties even for the Δ^1_3 level.

5. Relativizing Ω and uniform stationarity

In light of observation 3.1, it is natural to define

Definition 5.1. ω_1 is *Mahlo to reals* if for every real a, ω_1 is Mahlo in $L[a]$.

and to ask:

Question 5.2. If Ω is stationary, does it follow that ω_1 is Mahlo to reals?

However, it will turn out that this is not the case. Lemma 8.2 shows that the stationarity of Ω does not even imply that ω_1 is inaccessible to reals.

Of course, it would be a beautiful conclusion to make, and the fact that one cannot might indicate that the assumption is not the right one. A slight strengthening of the assumption enables us to draw the desired conclusion.

Definition 5.3. For $A \subseteq \mathrm{On}$, let

$$\Omega_A = \{\omega_1^{L[A,x]} \mid x \subseteq \omega\} \cap \omega_1.$$

Then Ω is *uniformly stationary* if for every $a \subseteq \omega$, Ω_a is stationary in ω_1.

Corollary 5.4. *If Ω is uniformly stationary, then ω_1 is Mahlo to reals.*

Proof. The proof of observation 3.1 relativizes. □

Another way to rephrase the above question would be:

Question 5.5. If Ω is stationary and ω_1 is inaccessible to reals, does it follow that ω_1 is Mahlo to reals?

Again, the answer is no, as lemma 8.3 will show.

Let us carry out the steps to arrive at the various characterizations of Ω for its relativized forms. First, let's define:

Definition 5.6. For $a \subseteq \mathrm{On}$, let

$$\bar{\Omega}_a = \{\omega_1^{L[b]} \mid a \in L[b] \models \text{“}b \text{ is reshaped”}\}$$

and

$$\bar{\Omega}'_a = \{\omega_1^{L[c]} \mid a \in L[c] \text{ and } c \subseteq \mathrm{On}\}$$

Observation 5.7. *For $a \subseteq \omega$, $\Omega_a \subseteq \bar{\Omega}_a \subseteq \bar{\Omega}'_a$.*

Proof. The proof of observation 2.5 relativizes. □

Theorem 5.8. *If ω_1 is inaccessible to reals, then for any real a, $\bar{\Omega}'_a \subseteq \bar{\Omega}_a = \Omega_a$.*

Proof. For the first inclusion, proceed as in the proof of the unrelativized version of the theorem. So let $\rho = \omega_1^{L[c]} \in \bar{\Omega}'_a$, for some $c \subseteq \mathrm{On}$ with $a \in L[c]$. Working inside $L[c]$, for every $\alpha < \omega_1^{L[c]}$, let $f^\alpha : \omega \longrightarrow \alpha$ be a surjection. Let $\bar{c} \subseteq \omega_1^{L[c]}$ code this sequence of functions, and let $c' = a \oplus \bar{c}$. Then $c' \subseteq \omega_1^{L[c]} = \omega_1^{L[c']} = \rho$. Let $b \in \mathrm{V}$ be generic for the reshaping forcing $\mathbb{P}$ with respect to c'. Since b codes c' and c' codes a, and the forcing preserves ω_1, it follows that $\rho = \omega_1^{L[b]}$, and $a \in L[b]$, so that $\rho \in \bar{\Omega}$.

The argument for the second inclusion is as in the proof of theorem 2.6. □

Theorem 5.9. *Assume there is no reshaped set. Then ω_1 is inaccessible to reals and Ω is uniformly stationary.*

Proof. It was already shown that ω_1 is inaccessible to reals. To see that Ω is uniformly stationary, fix $a \subseteq \omega$. It suffices to show that $\bar{\Omega}_a$ is stationary. So let C be club in ω_1. As in the proof of theorem 4.1, I will construct a sequence $s : \rho \longrightarrow 2$ by recursion, so that

(1) if $\alpha < \rho$ is a limit ordinal, then $\omega_1^{L[s\restriction\alpha]} > \alpha$,
(2) if $\beta < \alpha < \rho$, β and α limits, then $(\omega_1^{L[s\restriction\beta]}, \omega_1^{L[s\restriction\alpha]}) \cap C \neq \emptyset$.
(3) $\omega_1^{L[s]} = \rho$.

The point is that there is some freedom as to how to start the sequence s. I want to set $s \restriction \omega = a$. The recursive definition onward proceeds as in the proof of theorem 4.1.

Since there is no reshaped set, the construction breaks down at a countable stage ρ, which has to be a limit point of C, as before, and so, $\rho \in C$. So, letting $b := \{\alpha < \rho \mid s(\alpha) = 1\}$, it follows that b is reshaped in $L[b]$, and since $b \restriction \omega = a$, it is clear that $a \in L[b]$. So by 3., $\rho = \omega_1^{L[b]} \in \bar{\Omega}_a$. So $\rho \in \bar{\Omega}_a \cap C$. □

6. Local Mahloness

Definition 6.1. ω_1 is Mahlo at $a \subseteq \omega$ if $\{\alpha < \omega_1 \mid \alpha \text{ is inaccessible in } L[a]\}$ is stationary in ω_1. ω_1 locally Mahlo if ω_1 is Mahlo at a, for every real a.

Theorem 6.2. *The following are equivalent, for a real $a \subseteq \omega$:*

(1) Ω_a *is stationary.*
(2) ω_1 *is Mahlo at* a.

Proof. To see the implication from 1. to 2, note that $\Omega_a = \{\omega_1^{L[a,x]} \mid x \subseteq \omega\}$ consists of countable ordinals that are regular in $L[a]$ and is stationary. Intersecting this set with the club set of countable ordinals that are limit cardinals in $L[a]$ produces a stationary set of ordinals that are inaccessible in $L[a]$. So ω_1 is Mahlo at a.

For the converse, assume $\kappa = \omega_1^{\mathrm{V}}$ is Mahlo at $a \subseteq \omega$. To show that Ω_a is stationary, let $C \subseteq \omega_1$ be club. By Mahloness at a, let $\alpha \in C$ be inaccessible in $L[a]$. Since ω_1 is not assumed to be inaccessible to reals, one has to go through the individual steps to find a real r such that $a \in L[r]$ and $\alpha = \omega_1^{L[r]}$. First, force with $\mathrm{Col}(\omega, {<}\alpha)$ over $L[a]$. Since $\alpha < \omega_1$ and ω_1 is Mahlo at a, there is an $L[a]$-generic g in V for that forcing, because it only has countably many dense sets that belong to $L[a]$. Clearly, $g' = a \oplus g$ can be viewed as a subset of α, and $\omega_1^{L[g']} = \alpha \in C$. Moreover, since g arose from a small forcing over $L[a]$ with respect to its Mahlo cardinal ω_1^{V}, κ is still Mahlo in $L[g']$. Working inside $L[g']$, $\alpha = \omega_1$ and $L_{\omega_2}[g'] = H_{\omega_2}$, so the reshaping forcing can be used to add a reshaped set over $L[g']$, reaching $L[b]$, where κ is still Mahlo, again, because the reshaping forcing is small with respect to κ (and there is a generic in V because $\kappa = \omega_1$). Then, the coding forcing can be applied over $L[b]$, resulting in a model $L[r]$, where $r \subseteq \omega$, $a \in L[r]$ and $\alpha = \omega_1^L[r]$ (again, r can be found in V because $\kappa = \omega_1$. So $\alpha \in \Omega_a \cap C$. □

Corollary 6.3. *The following are equivalent:*

(1) Ω *is uniformly stationary.*
(2) ω_1 *is locally Mahlo.*

Let κ^M be the least inaccessible cardinal of the inner model M, if it has an inaccessible cardinal. Since the uniform stationarity of Ω implies that ω_1 is Mahlo to reals, it follows that $\kappa^{L[a]}$ is defined for every real a. So another very natural variant of Ω would be

$$\Omega_{\text{i.a.}} = \{\kappa^{L[a]} \mid a \subseteq \omega\} \cap \omega_1.$$

Basically, "the smallest uncountable cardinal" in the definition of Ω is replaced with "the smallest inaccessible cardinal" in the definition of $\Omega_{\text{i.a.}}$. Clearly, this definition can be relativized from $\Omega_{\text{i.a.}}$ to $(\Omega_{\text{i.a.}})_a$, just like Ω was relativized to Ω_a.

Similar arguments as the ones used before show that the stationarity of Ω is equivalent to the stationarity of $\Omega_{\text{i.a.}}$. Clearly, if $\Omega_{\text{i.a.}}$ is stationary, then so is Ω, because every inaccessible cardinal of some $L[a]$ is the ω_1 of some $L[b]$, by collapsing the inaccessible to be ω_1, adding a reshaped set, and coding by a real. Vice versa: If Ω is stationary, then this means that the countable ordinals that are regular cardinals in L form a stationary set. Since the limit cardinals of L are a club in V, the set of inaccessible cardinals of L is stationary in V. Every inaccessible cardinal α of L can be forced to be the smallest inaccessible cardinal (by first shooting a club C through the non-inaccessible cardinals of L below α, so that in that forcing extension, the set of inaccessibles below α is not stationary, and then collapsing each inaccessible $\gamma < \alpha$ to $\sup(C \cap \gamma)^+$). So every such α is equal to $\kappa^{L[g]}$, for some $g \subseteq \alpha$. Now one can force over $L_\delta[g]$, where δ is the next inaccessible cardinal of $L[a]$ above α, say, to code g by a real r (see [Fri00, Section 4.3]). All of this can be done within V, and the argument relativizes, showing that Ω_a is stationary iff $(\Omega_{\text{i.a.}})_a$ is stationary, for $a \subseteq \omega$. I take this robustness of these concepts as an indication for their naturalness.

The following diagram summarizes the implications between the different concepts (excluding $\Omega_{\text{i.a.}}$):

FIGURE 1. Diagram of implications

$$\begin{tikzcd}
\text{No reshaped set} \arrow[d] & & & \\
\Omega\text{ is uniformly stationary} \arrow[r, leftrightarrow] \arrow[d] & \omega_1\text{ is locally Mahlo} \arrow[r] \arrow[d] & \omega_1\text{ is Mahlo to reals} \arrow[r] \arrow[d] & \omega_1\text{ is inaccessible to reals} \arrow[d] \\
\Omega_a\text{ is stationary} \arrow[r, leftrightarrow] & \omega_1\text{ is Mahlo at }a \arrow[r] & \omega_1\text{ is Mahlo in }L[a] \arrow[r] & \omega_1\text{ is inaccessible in }L[a]
\end{tikzcd}$$

7. Assumptions implying versions of the stationarity of Ω

It was shown in lemma 4.4 that $\mathsf{MA}_{\omega_1}(\sigma-\text{centered})+\omega_1$ is inaccessible to reals implies that there is no reshaped set. I want to explore other natural assumptions with the same consequence.

Lemma 7.1. *Assume that ω_1 is inaccessible to reals after forcing with any σ-centered forcing of size at most ω_1. Then there is no reshaped set.*

Proof. Assume the contrary. Let b be reshaped. Let G be $\mathbb{P}_b$-generic over V. Then G adds a real x such that $b \in L[x]$, in such a way that $\omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[G]}$ (it is easy to see that $\mathbb{P}_b$ is σ-centered also in V, not only in $L[b]$, after looking at its definition). Since $\mathbb{P}_b$ has size ω_1 and is σ-centered, it follows by our absoluteness assumption that in $\mathrm{V}[G]$, ω_1 is still inaccessible to reals. But $\omega_1^{L[x]} = \omega_1^{L[b]} = \omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[G]}$, a contradiction. □

Lemma 7.2. *$\utilde{\Sigma}^1_4$-absoluteness for σ-centered forcing, together with the assumption that ω_1 is inaccessible to reals, implies that ω_1 is inaccessible to reals in every forcing extension by a σ-centered forcing. So it implies that there is no reshaped set.*

Proof. It can be expressed by a Π^1_4 statement that ω_1 is inaccessible to reals. The conclusion follows from the previous lemma. □

According to [BF01, Remark after proof of thm. 6], it follows from [BB04] that if κ is Mahlo and G is generic for $\mathrm{Col}(\omega,{<}\kappa)$, $\mathrm{V}[G]$ satisfies $\utilde{\Sigma}^1_4$-absoluteness for σ-centered forcing. Since in $\mathrm{V}[G]$, ω_1 is inaccessible to reals, this implies that there is no reshaped set in $\mathrm{V}[G]$. This is another way to see that the consistency strength of the absence of a reshaped set is a Mahlo cardinal.

Lemma 7.3. *$\utilde{\Sigma}^1_4$-absoluteness for c.c.c. forcing implies that there is no reshaped set.*

Note: The consistency strength of this absoluteness is a weakly compact cardinal, by [BF01].

Proof. $\utilde{\Sigma}^1_4$-absoluteness for c.c.c. forcing implies that ω_1 is inaccessible to reals, by [BF01, Thm. 7]. The rest follows from the previous lemma. □

Lemma 7.4. *If every subset of ω_1 has a sharp, then there is no reshaped set.*

Proof. It follows from the assumption of the lemma that ω_1 is inaccessible to reals in every ccc forcing extension by a forcing of size at most ω_1, because if x is a real of such an extension, there is a name $\dot{x}$ for x which can be viewed as a subset of ω_1, so its sharp exists in V, which means that in the extension, every real has a sharp. This implies, of course, that ω_1 is inaccessible to reals. So by lemma 7.1, there is no reshaped set. A different, more direct way to prove the present lemma was pointed out by the referee: clearly, under our assumption, no bounded subset of ω_1 can be reshaped. But if $b \subseteq \omega_1$ is unbounded and $b^{\#}$ exists, then there is a club of $\alpha < \omega_1$ such that there is an elementary embedding $j : L[b \cap \alpha] \longrightarrow L[b]$, and for such an α, $j(b \cap \alpha) = b$, so the critical point of j is at most α. So $L[b \cap \alpha]$ cannot see that α is countable. So b is not reshaped. □

Theorem 7.5. *If there is a precipitous ideal on ω_1, then there is no reshaped set.*

Proof. If there is a precipitous ideal on ω_1, then every subset of ω_1 has a sharp, by the following argument (supplied by the referee): Let G be generic for $\mathcal{P}(\omega_1) \setminus I$, where I is a precipitous ideal on ω_1. Let $j : \mathrm{V} \longrightarrow_G M$. Firstly, it is immediate that every real has a sharp, because j can be restricted to $L[a]$, for every real a. But then, if $A \subseteq \omega_1$ is unbounded, then in M, $A = j(A) \cap \omega_1^{\mathrm{V}}$ is a subset of a countable ordinal, and hence coded by a real. Since in V, every real has a sharp, the same is true in M, by elementarity, and so, $A^\#$ exists in M. So $A^\#$ exists in $\mathrm{V}[G]$, and since the sharp of A cannot be added by forcing, $A^\#$ exists in V. So the lemma follows from Lemma 7.4.

Another direct way to see it is as follows: suppose $b \subseteq \omega_1^{\mathrm{V}}$ was reshaped. Then in M, $j(b)$ would be reshaped. So in M, it would be true that for every $\alpha < \omega_1 = j(\omega_1^{\mathrm{V}})$, $L[j(b) \cap \alpha]$ sees that α is countable. In particular, this would be true for $\alpha = \omega_1^{\mathrm{V}}$. But $j(b) \cap \omega_1^{\mathrm{V}} = b$, so $L[b]$ would see that ω_1^{V} is countable. But $b \in \mathrm{V}$, so this cannot be. □

Theorem 7.6. *If the nonstationary ideal on ω_1 is precipitous, then there is no reshaped set, and ω_1 is inaccessible to reals, yet there is a forcing that adds a real x such that $\omega_1^{\mathrm{V}} = \omega_1^{L[x]}$.*

Proof. By theorem 7.5, there is no reshaped set, ω_1 is inaccessible to reals, and Ω is stationary. The forcing $\mathbb{P}$ is the part of $\mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1}$ below Ω: Let G be $\mathbb{P}$-generic, and let $j : \mathrm{V} \longrightarrow_G M$ be the corresponding generic embedding and ultrapower. Since $\Omega \in G$, $\omega_1^{\mathrm{V}} \in j(\Omega)$, so in M, there is a real x such that $\omega_1^{\mathrm{V}} = \omega_1^{L[x]}$. □

Recall that if it is consistent that Ω is stationary, then it is consistent that it contains a club, because one can shoot a club through Ω by a σ-distributive forcing (see Theorem 4.5). It is interesting that some stronger assumptions actually imply that Ω contains a club, though. For example:

Observation 7.7. *If the universe is closed under sharps, then Ω contains a club.*

Proof. Clearly, Ω is stationary, by lemma 7.4. If Ω did not contain a club, then one could shoot a club through its complement. In the corresponding forcing extension, the universe would still be closed under sharps, which implies that Ω is stationary, a contradiction. □

8. Non-implications

The purpose of this section is to prove that the diagram of implications (figure 1) is complete, i.e., that no arrows can be reversed, and that no other implications hold (which don't result from the diagram via transitivity). For the results in this section to make sense, one must assume the consistency of ZFC with the existence of a Mahlo cardinal. For some non-implications, I will take the liberty to assume the consistency of slightly stronger large cardinals.

Lemma 8.1. *If it is consistent that Ω is uniformly stationary, then it is consistent that Ω is uniformly stationary but there is a reshaped set.*

Proof. Under the assumption, it is consistent that κ is Mahlo in L. Let G be $\mathrm{Col}(\omega, <\kappa)$-generic over L. In $L[G]$, Ω is uniformly stationary, by lemma 3.3 (or rather, by the proof of the lemma, which relativizes to show that Ω_a is stationary in $L[G]$, for every $a \subseteq \omega$ in $L[G]$.) Note that G can be viewed as a subset of ω_1 (in $L[G]$), so that in $L[G]$, $L_{\omega_2}[G] = H_{\omega_2}^{L[G]}$. By theorem 2.3, it is possible to force a reshaped set under this assumption, using a forcing which is $<\omega_1$-distributive and stationary set preserving. Let H be generic for this forcing over $L[G]$. Since the forcing is $<\omega_1$-distributive, $\Omega_a^{L[G][H]} = \Omega_a^{L[G]}$, and since it is stationary set preserving, $\Omega_a^{L[G]}$ is stationary in $L[G][H]$, so the proof is complete. $\square$

Lemma 8.2. *If it is consistent that Ω_a is stationary, then it is consistent that Ω_a is stationary, yet ω_1 is not inaccessible to reals.*

Proof. By the previous lemma, it is consistent that Ω_a is stationary yet there is a reshaped set. If b is reshaped, then forcing with $\mathbb{P}_b$ over V adds a real x such that $b \in L[x]$. Since $\mathbb{P}_b$ is c.c.c., $\omega_1^{\mathrm{V}[x]} = \omega_1^{\mathrm{V}}$, and so, $\omega_1^{L[x]} = \omega_1^{\mathrm{V}[x]}$, so ω_1 is not inaccessible to reals in $\mathrm{V}[x]$. But of course, Ω_a^{V} is stationary in $\mathrm{V}[x]$, as $\mathbb{P}_b$ preserves stationary sets, and trivially, $\Omega_a^{\mathrm{V}} \subseteq \Omega_a^{V[x]}$. So $\Omega_a^{\mathrm{V}[x]}$ is stationary in $\mathrm{V}[x]$. $\square$

The previous lemma also answers question 5.2 in the negative. It does not, however, answer question 5.5, which is whether the stationarity of Ω, together with the inaccessibility of ω_1, implies that ω_1 is Mahlo to reals. It is very natural to ask this question, because the inaccessibility of ω_1 is somehow the background assumption that makes everything work more smoothly.

Lemma 8.3. *If* ZFC *is consistent with the existence of $0^\#$ and an inaccessible, then it is consistent that ω_1 is inaccessible to reals, Ω is stationary, but ω_1 is not Mahlo to reals.*

Proof. Assume $\mathrm{V} = L[0^\#]$ and κ is inaccessible, but not Mahlo. Let $G \subseteq \mathrm{Col}(\omega, <\kappa)$ be generic. Then in $M = \mathrm{V}[G]$, $\kappa = \omega_1$ is inaccessible to reals, so that $\Omega^M = \{\alpha < \kappa \mid \alpha \text{ is regular in } L\}$. Note that every cardinal of V is inaccessible in L, so Ω^M contains a club. $\omega_1 = \kappa$ is not Mahlo in $L[0^\#]$, so the real $0^\#$ witnesses that ω_1 is not Mahlo to reals in $\mathrm{V}[G]$. $\square$

In particular, the stationarity of Ω does not imply the uniform stationarity of Ω (since that would imply that ω_1 is Mahlo to reals), even if one assumes that ω_1 is inaccessible to reals.

Lemma 8.4. *It is consistent that ω_1 is Mahlo to reals but not locally Mahlo.*

Proof. Suppose κ is Mahlo in L, and let G be $\mathrm{Col}(\omega, <\kappa)$-generic over L. In $L[G]$, Ω is uniformly stationary, so ω_1 is both Mahlo to reals and locally Mahlo. In $L[G]$, consider the set $I = \{\alpha < \kappa \mid \alpha \text{ is inaccessible in } L\}$. M does not contain a club in $L[G]$, since the collapse is κ-c.c., so if it did contain a club, that club would have a club subset in L, but clearly, L has no club subset of I. So the complement of I is stationary, which means that one can shoot a club C through the complement of I over $L[G]$. Adding the club is a σ-distributive forcing notion, so in $L[G, C]$, ω_1 is still κ, and so, since no reals were added, ω_1 is still Mahlo to reals. But of course, I is not stationary anymore, so that ω_1 is not locally Mahlo anymore (local Mahloness fails in the strongest possible sense, namely, the inaccessible cardinals of L below ω_1 don't form a stationary set!). $\square$

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