PROBLEM SET 9 FOR MATH 71200
- SET THEORY AND LOGIC -
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Problem 1:
Work in ZF^-. Show that V_ω is the unique set u that has the following properties:

(1) u is transitive.
(2) u is closed under finite subsets. That is, if $a \subseteq u$ is finite, then $a \in u$.
(3) Every set in u is finite.

Problem 2:
Work in ZF^-. For $n \in \omega$, let u_n be the unique finite set such that $n = \sum_{i \in u_n} 2^i$. Define a binary relation $\tilde{\in}$ on ω by letting

$$m \tilde{\in} n \iff m \in u_n.$$

(a) Show that $\langle \omega, \tilde{\in} \rangle$ is extensional and well-founded.
(b) Let $\pi : \langle \omega, \tilde{\in} \rangle \overset{\sim}{\longleftrightarrow} \langle u, \in \restriction u \rangle$, where u is transitive (as given by Mostowski's isomorphism theorem). Show that $u = \mathrm{V}_\omega$ (using Problem 1).

Problem 3:
Let's consider the form of properties (1)-(3) of Problem 1, one cardinality higher:

(1') u is transitive.
(2') u is closed under countable subsets. That is, if $x \subseteq u$ is countable, then $x \in u$.
(3') Every set in u is at most countable.

Denote by $\mathsf{TC}(x)$ the *transitive closure of x,* that is the least (with respect to inclusion) set y such that $x \subseteq y$ and y is transitive. Working in ZFC^-, show:

(a) A set u has properties (1')-(3') iff $u = HC$, where

$$HC = \{x \mid \mathsf{TC}(x) \text{ is at most countable}\}$$

is the collection of *hereditarily countable sets.*
(b) $HC \neq \mathrm{V}_{\omega_1}$. So Problem 1 does not generalize in this sense.
(c) Letting $HF = \{x \mid \mathsf{TC}(x) \text{ is finite}\}$ (the collection of *hereditarily finite sets*, it follows that $HF = \mathrm{V}_\omega$. So Problem 1 generalizes in the sense that (1)-(3) characterize HF and (1')-(3') characterize HC.

Please submit your homework by email, as a pdf file created with LaTeX, by 4/7/2019.