PROBLEM SET 8 FOR MATH 71200 - SET THEORY AND LOGIC - SPRING 2019

DR. GUNTER FUCHS

Problem 1:
Work in ZF^-. Consider the following principles:
Compactness: if Σ is a set of formulas (in any language) such that every finite subset of Σ has a model, then Σ has a model.
Completeness: if $\Sigma \cup \{\varphi\}$ is a set of formulas (in any language) and $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.
The *Linear Order Principle*: every set can be linearly ordered (i.e., for any set u, there is a binary relation $r \subseteq u \times u$ such that $\langle u, r\rangle$ is a linear order).
The *Ultrafilter Lemma*: every filter in a Boolean algebra is contained in an ultrafilter.

All of these principles are provable in ZFC. We proved the special case of Completeness where Σ is countable in ZF^-. Show the following:

(1) Compactness and Completeness are equivalent.
(2) Compactness implies the Linear Order Principle.
(3) The Linear Order Principle implies the following weak form of the axiom of choice: every set of *finite* nonempty sets has a choice function.
(4) Compactness implies implies the Ultrafilter Lemma.

Note: If ZF is consistent, then so is the theory ZF, together with the assertion that there exists a set $\{a_n \mid n < \omega\}$ of *two-element* sets that has no choice function. By (1), (2) and (3), Compactness and Completeness fail in any model of this theory.

Problem 2:
Optional! Show in ZF^- that the Ultrafilter Lemma implies Completeness. For the argument, you can use what's known as the Henkin construction. Sketching the proof is enough!

Problem 3:
Find a formula that defines the graph of the function $f : \langle x, y\rangle \mapsto x + y$ over $\mathfrak{N}$, yet is not a functional representation of f. Show that your formula works.
Hint: You can use Problem 2 of Homework set 7 here.

Problem 4:
Show that the following functions are representable:

(1) The projections $\pi^n_l : \omega^n \longrightarrow \omega$, for $l < n < \omega$, which are defined by
$$\pi^n_l(x_0, \ldots, x_{n-1}) = x_l,$$
(2) the functions $m \mapsto m+1$, $\langle m, n\rangle \mapsto m+n$, $\langle m, n\rangle \mapsto m\cdot n$ and $\langle m, n\rangle \mapsto m^n$,
(3) the constant functions $c^n_k : \omega^n \longrightarrow \omega$, for $k, n < \omega$, which are defined by
$$c^n_k(m) = k,$$
(4) the function $m \mapsto [m : 2]$, the integer part of $m/2$.

Please submit your homework by email, as a pdf file created with LaTeX, by 3/31/2019.