PROBLEM SET 7 FOR MATH 71200 - SET THEORY AND LOGIC - SPRING 2019

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For this problem set, work in the theory ZF.

The language of number theory with exponentiation consists of the constant symbol 0, the binary relation symbol $<$, the unary function symbol S, and the binary function symbols $+, \cdot$ and E. Using infix notation for $+, \cdot$ and $<$, we will work with the following set of axioms, called A_E (for "arithmetic with exponentiation"):

(S1) $\forall v_0 \quad \neg S(v_0) = 0$
(S2) $\forall v_0 \forall v_1 \quad (S(v_0) = S(v_1) \to v_0 = v_1)$
(L1) $\forall v_0 \forall v_1 (v_0 < S(v_1) \leftrightarrow (v_0 < v_1 \vee v_0 = v_1))$
(L2) $\forall v_0 \quad \neg v_0 < 0$
(L3) $\forall v_0 \forall v_1 (v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0)$
(A1) $\forall v_0 \quad v_0 + 0 = v_0$
(A2) $\forall v_0 \forall v_1 \quad (v_0 + S(v_1) = S(v_0 + v_1))$
(M1) $\forall v_0 \quad v_0 \cdot 0 = 0$
(M2) $\forall v_0 \forall v_1 \quad (v_0 \cdot S(v_1) = (v_0 \cdot v_1) + v_0)$
(E1) $\forall v_0 \quad E(v_0, 0) = S(0)$
(E2) $\forall v_0 \forall v_1 \quad E(v_0, S(v_1)) = E(v_0, v_1) \cdot v_0$

Problem 1:
On problem set number 3, we defined ordinal addition, using the Recursion Theorem. As hinted at in class, let's define ordinal multiplication by

$$\begin{aligned} \alpha \cdot 0 &= 0, \\ \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha, \\ \alpha \cdot \lambda &= \bigcup \{\alpha \cdot \beta \mid \beta < \lambda\}, \text{ for limit } \lambda. \end{aligned}$$

Similarly, we can define ordinal exponentiation by

$$\begin{aligned} \alpha^0 &= 1, \\ \alpha^{\beta+1} &= \alpha^\beta \cdot \alpha, \\ \alpha^\lambda &= \bigcup \{\alpha^\beta \mid \beta < \lambda\}, \text{ for limit } \lambda. \end{aligned}$$

Show (in ZF) that there arbitrarily large ordinals γ that are closed under addition, multiplication and exponentiation (the smallest such is ω, and the next one is called ϵ_0 - can you describe it?). If $\gamma > 0$ is closed under these operations, then show that

$$\langle \gamma, 0, S{\restriction}\gamma, +{\restriction}(\gamma \times \gamma), \cdot{\restriction}(\gamma \times \gamma), E{\restriction}(\gamma \times \gamma), <{\restriction}\gamma \rangle \models A_E$$

Here, $E(\alpha, \beta) = \alpha^\beta$ and S is the usual ordinal successor operation.

Problem 2:
Prove or disprove:

$$A_E \vdash \forall v_0 (v_0 \neq 0 \longrightarrow \exists v_1 (v_0 = S(v_1))).$$

Please submit your homework by email, as a pdf file created with latex, by 3/24/2019.