PROBLEM SET 4 FOR MATH 71200 - SET THEORY AND LOGIC - LOGIC I
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The principle of Dependent Choices (DC) is a weak form of the axiom of choice stating that whenever $\langle u, r\rangle$ is a binary system with the property that for every $y \in u$, there is an $x \in u$ such that xry, it follows that there is an infinite decreasing chain in $\langle u, r\rangle$, that is, a function $f : \omega \longrightarrow u$ such that for all $n \in \omega$, $f(n+1) r f(n)$.

Problem 1:
Prove in ZF + DC that a binary system $\langle u, r\rangle$ fails to be well-founded iff there is an infinite decreasing chain in $\langle u, r\rangle$, that is, a function $f : \omega \longrightarrow u$ such that for all $n \in \omega$, $f(n+1) r f(n)$. Conclude that König's Lemma holds in ZF + DC: every infinite, locally finite tree has an infinite branch. Actually, an even weaker form of choice suffices for the latter: every countable set of finite, nonempty sets has a choice function.

Problem 2:
The countable Löwenheim-Skolem Theorem says that every infinite model of a countable first order language language has a countable elementary submodel. Show in ZF that the countable Löwenheim-Skolem Theorem is equivalent to DC.

Hint: To show that DC implies the countable Löwenheim-Skolem Theorem, given an infinite model M, let u consist of one-to-one functions $f : n \longrightarrow |M|$, for some $n < \omega$, and define frg if $\mathrm{dom}(f) > \mathrm{dom}(g)$ and, fixing an enumeration of the formulas in the language of M, we have that for every $i < \mathrm{dom}(g)$, if $\varphi_i(w, v_0, \ldots, v_{l-1})$ is the i-th formula, and $a_0, \ldots, a_{l-1} \in \mathrm{ran}(g)$, if $M \models \exists w\ \varphi_i[a_0, \ldots, a_{l-1}]$, then there is a $b \in \mathrm{ran}(f)$ such that $M \models \varphi_i[b, a_0, \ldots, a_{l-1}]$.

For the converse, note that any countable set is automatically well-ordered.

Problem 3:
Show in ZF^{--}+ Infinity: if $R \subseteq \mathrm{V}^2$ is a set-like relation, then for every set u, there is a minimal (with respect to inclusion) set v that's R-closed and contains u. Show also that this result does not hold in ZF without the axiom of infinity, and that it does not hold if R is not set-like.

Problem 4:
Let's say that a relation R is *well-founded with respect to sets* if every nonempty *set* a has an R-minimal element. And let's say R is *well-founded with respect to classes* for the scheme expressing that every nonempty class has an R-minimal element.

(1) Show that ZF^{--} + Infinity+"R is set-like and well-founded with respect to sets" proves the scheme "R is set-like and well-founded with respect to classes". So in ZF^{--} + Infinity, this scheme can be expressed by a single formula. In particular, the recursion theorem can be expressed in this theory.
(2) Show that in ZF, if R is well-founded with respect to sets, then R is well-founded with respect to classes (that is, every nonempty class has an R-minimal element - note that it is not assumed here that R is set-like).

Please submit your homework by email, as a pdf file created with latex, by 3/3/2019.